The Dynamics of Social Influence*

Bary S. R. Pradelski†

This version: August 9, 2017

Abstract. Individual behavior such as choice of fashion, adoption of new products, and selection of means of transport is influenced by taking account of others' actions. We study social influence in a heterogeneous population and analyze the behavior of the dynamic processes. We distinguish between two information regimes: (i) agents are influenced by the adoption ratio, (ii) agents are influenced by the usage history. We identify the stable equilibria and long-run frequencies of the dynamics. We then show that the two processes generate qualitatively different dynamics, leaving characteristic 'footprints'. In particular, (ii) favors more extreme outcomes than (i).

JEL classifications: C62, C70, D70, D83, O33, Z13

Keywords: social influence, imitation, equilibrium selection

1 Introduction

A fundamental question about aggregate behavior of groups is how shifts between seemingly stable states occur almost instantaneously after long lags of low fluctuation. Social influence has been found to play an important role in such transitions. Social influence describes the process in which individuals are influenced by the behaviors of others in a group. For example, the emergence of fashions is well documented to be driven by social

*I thank Peyton Young for his guidance and support throughout this research. I thank Johannes Ruf for framing discussions at an early stage of this project. Further, I thank Sergiu Hart, Dirk Helbing, Willemien Kets, Michael Maes, Heinrich Nax, Bill Sandholm, Pierre Tarrès, anonymous referees, and the participants at the 25th International Conference on Game Theory at Stony Brook, the Transatlantic Theory Workshop, Paris (2014), the European Econometric Society Meeting, Geneva (2016), and the seminar at the Department of Computational Social Science at ETH Zürich for helpful comments. I would like to acknowledge the generous support of the Oxford-Man Institute of Quantitative Finance, the Office of Naval Research grant # N00014-09-1-0751, the Air Force Office of Scientific Research grant # FA9550-09-1-0538, and the European Comission through the ERC Advanced Investigator Grant Momentum # 324247.

†ETH Zurich, Switzerland. Email: bpradelski@ethz.ch

influence. Other examples include stock market rallies and opinion extremization and polarization.

The purpose of this paper is to study two general families of dynamic processes of social influence dependent on the nature of information available. Agents update their actions at random points in time. Their decisions are influenced by the actions previously taken by other members of their group, where the strength of this influence varies across agents (heterogeneity). In some environments an agent chooses a durable action and adoption statistics are observable or public information. For example, the proportion of Apple's iPhone versus Samsung's smartphones can be obtained through publicly available sources. In such cases an agent essentially considers the adoption ratio, that is, an agent considers the proportion of iPhone versus Samsung smartphone owners. We denote this scenario by *Adoption Ratio*. In other environments, each agent chooses a non-durable action and the act of choosing is the critical information. For example, the proportion of smartphone owners using WhatsApp versus WeChat to communicate with each other depends on the repeated choice, rather than the fact that they have a given application installed on their device. Hence an agent considers the usage history, that is, an agent considers the historical proportion of times he was contacted with WhatsApp versus WeChat. We shall thus denote this scenario *Usage History*.[1]

Some environments are better described by one model and others by the other. Consider, as another example, the use of bicycles for the daily commute to work. People have different personal reasons to use a bicycle or another means of transport to commute to work, for example, distance to work, personal fitness, income, or environmental considerations. In addition, people are influenced by the behaviors of others and have a propensity to conform. Our study explores this latter influence while maintaining the fact that players are heterogeneous. On the one hand, the number of bicycle owners may be an important factor for an agent's decision (*Adoption Ratio*). On the other hand, the time series of choices, that is, how often bicycles are seen to be used to commute to work may well be an equally important factor (*Usage History*). Knowing which process is at work is relevant for interventions. On the one hand, if the observed process follows *Adoption Ratio* an incentive to purchase a bicycle would be the right intervention. Such an intervention was used in London, UK, where the *Cyclescheme* allows employees to purchase bicycles tax-free and thus save about half the cost. On the other hand, if the observed process follows *Usage History* an incentive should aim at increasing the frequency of choices. Again, such an intervention was used in London, UK, where the *Boris Bikes*[2] (a public bicycle hire scheme as found in many other cities) allows people to use public bicycles for free for the first 30 minutes (with a small annual subscription fee). In order to choose the most effective intervention it is instrumental to know which process of social influence, *Adoption Ratio* or *Usage History*, is at work.

The contribution of this paper is twofold. First, we identify the equilibria of the *Adoption Ratio* and *Usage History* process and study their stability in a stochastic environment.

[1]Note that this differentiation has not gone unnoticed in marketing departments. On the one hand, hardware providers such as Apple focus their reporting and marketing effort on the number of products sold (*Adoption Ratio*). On the other hand, software providers, such as WhatsApp, focus their analysis and messaging on the frequency of usage of their products (*Usage History*).

[2]Formally *Santander Cycles*.

Second, we find that the long-run behavior of these seemingly similar processes differs and we elaborate on the qualitative differences. We show that each process leaves a characteristic footprint – *Usage History* favors more extreme outcomes than *Adoption Ratio.*

2 Related literature

The discussion of social influence has a long history in economics, sociology and psychology (see, for example, Le Bon 1895, Trotter 1916, Keynes 1936, Hamilton 1971, Schelling 1971, Shiller 2000, Fehr and Hoff 2011).[3] Numerous applications have motivated the study of social influence, for example, political and social movements (Schelling 1978, Cabinet Office 2012), diffusion processes such as innovation adoption (Rogers 1962, Bass 1969, Meade and Islam 2006, Young 2009, Loeper et al. 2014, Newton and Sawa 2015), and financial herding (Scharfstein and Stein 1990, Banerjee 1992, Bikhchandani et al. 1992, Devenow and Welch 1996, Bouchaud 2013). Consequently there is broad experimental evidence for social influence. Asch (1955) conducted a series of enlightening experiments showing that a considerable proportion of subjects trust the majority over their own senses. More recently Salganik et al. (2006) show the effects of social influence in a study on music taste. Other experimental studies include voting and opinion polls (Cukierman 1991), human fertility (Bongaarts and Watkins 1996), diffusion of information technologies (Teng et al. 2002), household energy consumption (Schultz et al. 2007), mobile phones (de Silva et al. 2011), and bicycle usage for the work commute (Goetzke and Rave 2011).

Our model follows Schelling (1978, Chapter 3) and Granovetter (1978). They describe the class of *critical mass models* of social interaction. Schelling (1978) notes that "though perhaps not in physical and chemical reactions, in social reactions it is typically the case that the 'critical number' for one person differs from another's." Thus the *tipping value* determines, for each player, the critical mass of the aggregate information about the population's actions at which a player will 'tip over' from playing one action to another.[4] We shall study a threshold model where heterogeneous players repeatedly revise their binary action. Our model is in discrete time with asynchronous updating.

The processes we consider are Markovian. On the one hand, we use the concept of *stochastic stability* (see Foster and Young 1990, Kandori et al. 1993, Young 1993).[5] The idea is to study a perturbed version of the original process, such that the resulting Markov process is irreducible and ergodic and therefore the process has a unique stationary distribution. By letting the level of noise approach zero one can identify those states that will be observed in the long-run with a frequency bounded away from zero. On the other

[3]For further studies see Macy (1991), Dodds and Watts (2005), Horst and Scheinkman (2006), Bramoulle (2007), Lopez-Pintado and Watts (2008), Baddeley (2010), Young (2011), Babichenko (2013), Maes and Opp (2016).

[4]The heterogeneity of social influence has been confirmed in recent work, suggesting that cognitive factors influence the propensity to herding behavior (Dohmen et al. 2012).

[5]Peski (2010) extended the model of Kandori et al. (1993) to general networks. Our work provides an extension to heterogeneous players in a similar spirit to Easley and Kleinberg (2010) and Neary (2012).

hand, we also make use of recent work on reinforced random walks (Pinsky 2013). Pinsky analyzes a random walk on $\mathbb{Z}$ whose probability of moving left or right depends on the recent history. By an appropriate translation we find the unique limit proportion of play even though all states are stochastically stable and thus stochastic stability does not allow any selection result.

3 The model

We shall first introduce the general framework for analyzing social influence. Let $P = \{1, ..., p\}$, $p \in \mathbb{N}$ be the set of players. Let $A = \{0, 1\}$[6] be the actions available to each player $i \in P$. Let $u_i : [0, 1] \times A \to \mathbb{R}$ be the utility of agent $i \in P$ when observing the aggregate statistic about the society $s \in [0, 1]$ and playing action $a \in A$. We shall define two specific functional forms for s later but for now it suffices to think of some aggregate statistic about the players' actions. Suppose that the utility of an action is separable into a component arising from a player's inherent preference for an action and a component specifying the utility he derives from social conformity. After normalizing, let $\pi_i \in \mathbb{R}$ be player i's direct utility difference when playing action 1 over action 0. Further let $\rho_i \in \mathbb{R}^+$ be a player's *index of social conformity.* Finally, suppose that the impact of social influence is linear. A player's utility from playing action a is then given by

$$u_i(s, a) = \begin{cases} \pi_i + \rho_i s & \text{if } a = 1, \\ \rho_i(1 - s) & \text{if } a = 0. \end{cases} \tag{1}$$

This is a coordination game when s is increasing in the number of players playing 1.

Note that a player is indifferent between the two actions when $u_i(s, 1) = u_i(s, 0)$. That is, agent i is indifferent if and only if

$$s = \frac{\rho_i - \pi_i}{2\rho_i} =: \mu(i) \tag{2}$$

We shall call $\mu(i)$ player i's *tipping value.* If $s > \mu(i)$ player i wants to play 1 and if $s < \mu(i)$ he wants to play 0. A player with $\mu(i) \in (-\infty, 0)$ always prefers to play action 1 and a player with $\mu(i) \in (1, \infty)$ always prefers action 0. We shall make the simplifying assumption that for all players i, $\mu(i)$ is not a multiple of $1/p$ which will ensure that a player always has a unique best response. Given the list of different tipping values $\mu_1, ..., \mu_n$ (players may have the same tipping values, hence $n \leq p$) let q_j be the fraction of players with tipping value μ_j, that is $q_j = \frac{\sum_{i=1}^{p} \mathbf{1}_{\mu(i)=\mu_j}}{p}$ (for $j = 1, \ldots, n$).

Let $f_i : \mathbb{R}^2 \to [0, 1]$ be the *response function* for player i, specifying the probability to play action 1 given his utilities $u_i(s, 1) \in \mathbb{R}$ and $u_i(s, 0) \in \mathbb{R}$. Note that $1 - f_i(\cdot, \cdot)$ is the

[6] To avoid unnecessary notation we name actions such that their sum indicates the average action taken.

probability that i plays action 0. We shall initially consider a best-response model:

$$f_i = \begin{cases} 1 & \text{if } s > \mu(i), \\ 0.5 & \text{if } s = \mu(i), \\ 0 & \text{else.} \end{cases} \tag{3}$$

We study a discrete time process where in each period $t = 0, 1, 2, \ldots$ a unique player gets activated. In a given period t the activated player i will be called *active*. Define

$$\mathbf{1}_i^t = \begin{cases} 1 & i \text{ is active in } t, \\ 0 & \text{else.} \end{cases} \tag{4}$$

We write $act(t) \in P$ for the player who is active in period t.

Let $s(t) \in [0, 1]$ be the aggregate statistic about society at the beginning of period t. For each player i, let a_i^t be the action he plays at time t. Then for all $i \in P$

$$a_i^t = \mathbf{1}_i^t \cdot B^t[f_i(u_i(s(t), 1), u_i(s(t), 0))] + (1 - \mathbf{1}_i^t) \cdot a_i^{t-1} \tag{5}$$

for all $t \geq 1$, where $(B^t)_{t \in \mathbb{N}}$ is a family of independent Bernoulli random variables taking values in A.

Let $\bar{a}^t = \sum_{i=1}^{p} a_i^t / p \in [0, \frac{1}{p}, \ldots, 1]$ be the population's average action in period t. We consider two processes of social influence, arising from responding to different observations about society as discussed in the introduction:

- **Adoption Ratio.** The state at the beginning of a given period t is given by the action profile $\mathbf{a}^{t-1} = (a_i^{t-1})_{i \in P}$. An active player responds to the *Adoption Ratio*:

$$s^{AR}(t) = \bar{a}^{t-1} \tag{6}$$

- **Usage History.** The state at the beginning of a given period t is given by the last k actions, that is, $(a_{act(t-k)}^{t-k}, \ldots, a_{act(t-1)}^{t-1})$. An active player responds to the *Usage History* in the past k (constant) periods:[7,8]

$$s^{UH}(t) = \frac{\sum_{v=t-k}^{t-1} a_{act(v)}^{v}}{k} \tag{7}$$

When unambiguous we shall sometimes omit the specification of the time period.

To illustrate, suppose there are four players. Player 1 initially plays 1 and all other players play 0. Suppose we are in time step $t = 7$ and play unfolded as shown in table 1. For *Adoption Ratio* the observation in period $t = 7$ of prior adopters of action 1 is

[7] For *Usage History*, we need to define $s(t)$ differently for $t < k$. We can simply assume the average of the past t actions.

[8] Note that $s(t)$ includes the active player's action. This is reasonable when players are presented with the aggregate statistic, but the analysis also carries through if one excludes the active player's action from $s(t)$.

Table 1: Actions up to $t = 6$

time step	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$
active player	–	player 2	player 1	player 4	player 1	player 2	player 2
player 1	1		$\boxed{1}$		$\boxed{1}$		
player 2	0	1				$\boxed{1}$	$\boxed{1}$
player 3	0						
player 4	0			$\boxed{0}$			

Relevant actions for $s^{AR}(7)$ circled, relevant actions for $s^{AR}(7)$ boxed ($k = 5$).

$s^{AR}(7) = 50\%$ (the relevant actions are circled in Table 1). For *Usage History* (with $k = 5$) the observation in period $t = 7$ of the frequency of choices of action 1 is $s^{UH}(7) = 80\%$ (the relevant actions are boxed in Table 1). Note that in our example player 3's previous actions have no influence on the observed time series while player 1 and 2's are counted twice.

We analyze the two processes for several regimes of sampling and errors. Initially we will study the unperturbed best-response dynamic. We then consider a uniform action tremble. That is, there exists a small probability $\varepsilon > 0$ such that an activated player picks an action uniformly at random. The response function for player i then is

$$f_i = \begin{cases} 1 - \frac{\varepsilon}{2} & \text{if } s > \mu(i), \\ 0.5 & \text{if } s = \mu(i), \\ \frac{\varepsilon}{2} & \text{else.} \end{cases} \tag{8}$$

4 Analysis

We shall first state a definition and a simple lemma, the proof of which can be found in Appendix A.

Definition 1. *Given the population's average action $\bar{a}$, let Agg give the share of players for whom 1 is a best-response when observing $\bar{a} \in [0, 1]$:*

$$Agg : [0, 1] \to \left\{0, \frac{1}{p}, \frac{2}{p}, \dots, 1\right\} \tag{9}$$

$$Agg(\bar{a}) = \frac{1}{p} \sum_{i=1}^{p} \mathbf{1}_{\mu(i) < \bar{a}} \tag{10}$$

Lemma 2. *Agg has at least one fixed point. If x^* is a fixed point of Agg all players of the same type have the same best response in the associated game state and hence play the same action, that is*

$$x^* \in \{\bar{q}_k := \sum_{j=1}^{k} q_j \, : \, \mu_k < \bar{q}_k < \mu_{k+1}\}_{k=1,\dots,n}.[9] \tag{11}$$

We denote by $\mathbf{a}^*_\alpha \in \{\mathbf{a}^*_1, \dots, \mathbf{a}^*_l\}$ the action profiles where all players play a best response and $\bar{a}(\mathbf{a}^*_\alpha) = x^*_\alpha$ and $x^*_\alpha \in \{x^*_1, ..., x^*_l\}$ is a fixed point of Agg (in increasing order).

4.1 Adoption Ratio

In this section we shall consider the social influence process *Adoption Ratio.* An active player bases his decision on the number of current adopters (see Equation (6)).

$$s^{AR}(t) = \bar{a}^{t-1} = \frac{\sum_{i \in P} a_i^{t-1}}{p}$$

Since the proofs of the results in this section are standard when one is familiar with stochastic stability analysis, they are relegated to appendix B.

Theorem 3. *The unperturbed* Adoption Ratio *dynamic has at least one absorbing state. The absorbing states of the dynamic process coincide with the fixed points of Agg and each absorbing state is associated with exactly one fixed point of Agg and vice versa. The set of absorbing states which can be reached is dependent on the initial state (when multiple absorbing states exist).*

Theorem 3 is closely related to the convergence result in Babichenko (2013). His model, in our language, allows for negative social influence. He shows convergence to approximate Nash equilibrium if the observation of society is discretized, that is $\lfloor s^{AR} \rfloor_\delta$ for some $\delta > 0$. Babichenko (2013) also shows that his dynamic converges in $O(n \log n)$ time steps.

4.1.1 Perturbed dynamics

We now consider the perturbed process with a uniform error rate as introduced in Equation (8).

Theorem 4. *Suppose players have* uniform action trembles *in the* Adoption Ratio *dynamic. The stochastically stable states are those states in $\boldsymbol{a}^*_\alpha \in \{\boldsymbol{a}^*_1, \dots, \boldsymbol{a}^*_l\}$ which are associated with $x^*_\alpha \in \{x^*_1, ..., x^*_l\}$ of Agg that minimize*

$$\boldsymbol{a}^*_\alpha : \;\; \gamma_{\boldsymbol{a}^*_\alpha} = \min_{\beta=1,\dots,l} \gamma_{\boldsymbol{a}^*_\beta} \tag{12}$$

[9]Set $\mu_{n+1} = 1$ for completeness.

where $\gamma_{\boldsymbol{a}^*_\alpha}$ is the stochastic potential:

$$\gamma_{\boldsymbol{a}^*_\alpha} = \sum_{\beta=1}^{\alpha-1} r_{x^*_\beta, x^*_{\beta+1}} + \sum_{\beta=\alpha+1}^{l} r_{x^*_\beta, x^*_{\beta-1}} \tag{13}$$

with

$$r_{x^*_\beta, x^*_{\beta+1}} = \max_{x \in \left[x^*_\beta, x^*_{\beta+1}\right]} \{x - Agg(x)\} \cdot p + 1 \tag{14}$$

$$r_{x^*_{\beta+1}, x^*_\beta} = \max_{x \in \left[x^*_\beta, x^*_{\beta+1}\right]} \{Agg(x) - x\} \cdot p + 1 \tag{15}$$

For generic games there exists a unique long-term stable state.

4.2 Usage History

In this section we shall consider the social influence process *Usage History*. An active player bases his decision on the time series of choices (see Equation (7)):

$$s^{UH}(t) = \frac{\sum_{v=t-k}^{t-1} a^v_{act(v)}}{k}$$

Theorem 5. *The unperturbed* Usage History *dynamic has absorbing states if and only if* 0 *and/or* 1 *are fixed points of Agg or the best response of all players is independent of social influence. In the former case $all-0$ and/or $all-1$ are the unique absorbing states. In the latter case the unique absorbing state is the unique fixed point of Agg.*

Proof. We shall first show that if 0 (or 1) is a fixed point of Agg the corresponding state is absorbing. Suppose that 0 is a fixed point. Then there exists an observation s^* below which all players want to play 0. Suppose that $s(t) < s^*$ in period t. Now independent of who is selected in subsequent periods he plays action 0 and thus for all $T \geq 0$, $s(t+T) \leq s(t) < s^*$. Hence $all-0$ is an absorbing state of the dynamic. If 1 is a fixed point a similar argument applies.

Now if the best response of any player is independent of social influence there clearly exists an absorbing state, namely the state where every player plays the action he prefers (independent of social influence) which thus constitutes the unique fixed point of Agg.

Next suppose that there exists some player for whom social influence may change the best response. Suppose there exists an absorbing state where some players play 0 and some play 1. Suppose a player currently playing 0, say i, will be convinced to play 1 if a high enough proportion of the population plays 1 (social influence may change his best response). Then, with positive probability enough players who currently wish to play 1 are selected successively changing the historic action profile such that when i is next selected his best response is 1. This shows that there is no mixed-action absorbing state. $\square$

4.2.1 Perturbed dynamics

We now consider the perturbed process with a uniform error rate as introduced in Equation (8). Recall that q_j is the fraction of players with tipping value μ_j.

Theorem 6. *Suppose players have* uniform action trembles *in the* Usage History *dynamic.*

- *If the unperturbed dynamic has an absorbing state* 0 *(*1*) is stochastically stable if* $r_{0,1} \geq r_{1,0}$ *(*$r_{1,0} \geq r_{0,1}$*).*
- *Else, let* $\bar{q}_k = \sum_{j=1}^{k} q_j$*. For an action profile* $\boldsymbol{a}_\alpha^* \in \{\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_l^*\}$ *associated with fixed point* $x_\alpha^* \in \{x_1^*, ..., x_l^*\}$ *(in increasing order). Let*

$$\underset{j \in \{1,\ldots,n\}: \mu_j < \bar{q}_j < \mu_{j+1}}{\arg\max} \frac{1}{\bar{q}_j^{\mu_j}(1-\bar{q}_j)^{1-\mu_j}} \prod_{k=2}^{j} \left(\frac{\bar{q}_{k-1}}{1-\bar{q}_{k-1}} \right)^{\mu_k - \mu_{k-1}} = \{j_1, \ldots, j_\Delta\} \tag{16}$$

Then the limit proportion of play of action 1 *is given by*

$$\lim_{k\to\infty} \lim_{t\to\infty} \lim_{\varepsilon\to 0} \frac{\sum_{v=0}^{t} a_{act(v)}^{v}}{t} = \frac{\sum_{\delta=1}^{\Delta} \frac{1}{1-\bar{q}_{j_\delta}} \cdot q_{j_\delta}}{\sum_{\delta=1}^{\Delta} \frac{1}{1-\bar{q}_{j_\delta}}} \tag{17}$$

For generic games there exists a unique maximizer of Equation (16)*, say* j_**. The latter formula then reduces to:*

$$\lim_{k\to\infty} \lim_{t\to\infty} \lim_{\varepsilon\to 0} \frac{\sum_{v=0}^{t} a_{act(v)}^{v}}{t} = q_{j_*} \tag{18}$$

Proof. First, suppose the unperturbed dynamic has an absorbing state. If the unperturbed dynamic has only one absorbing state the result is trivial since the set of stochastically stable states is a subset of the set of recurrent classes which exactly constitutes the absorbing states (there are no non-unitary absorbing sets). Next we consider the case where the dynamic has two absorbing states, namely $all-0$ and $all-1$. Note that from any state where there exists at least one player whose best response is 0 and one player whose best response is 1 there exists a positive probability (independent of ε) to go to any other state. This holds since any value for s^{UH} occurs with strictly positive probability. To reach such a state from $all-0$ ($all-1$), $r_{0,1}$ ($r_{1,0}$) trembles are needed. Hence the minimizer is stochastically stable. This proves the first case of the theorem.

Now, suppose the unperturbed dynamic has no absorbing state. It follows that there is at least one player playing action 0 and one player playing action 1 independent of social influence and by the proof of the first case we thus have that the process is ergodic. Thus the uniform error does not change the limiting behavior of the dynamic and we can therefore set $\varepsilon = 0$. Now the result will follow from Pinsky (2013, Theorem 4). He studies a random walk on $\mathbb{Z}$ which at each point in time either takes one step to the right or one step to the left. Initially the probability of jumping one step to the right is $\bar{q}_1$ and of jumping to the left $1-\bar{q}_1$. If the proportion of jumps to the right in the last k moves is greater or equal to μ_j (and smaller than μ_{j+1}) then the probability of jumping to the right is $\bar{q}_j$ and of jumping to the left $1 - \bar{q}_j$. Pinsky studies the 'speed' of the process $\tilde{s} = \lim_{t\to\infty} \frac{X_t}{t}$ where X_t is the position on $\mathbb{Z}$ of the random walk at time t when starting at $X_t = 0$. He finds results according to the theorem above.

At time t, $\tilde{s}(\cdot)\cdot t$ is the number of times the process stepped to the right minus the number of times the process stepped to the left. Thus in $t - t\cdot\tilde{s}(\cdot)$ steps the process stepped equally often right as left. Hence the number of steps to the right is given by

$$\tilde{s}\cdot t + \frac{t - \tilde{s}\cdot t}{2} = \frac{\tilde{s}\cdot t + t}{2} = \sum_{v=0}^{t} a^v_{act(v)} \tag{19}$$

We thus have

$$\frac{\tilde{s}(\cdot) + 1}{2} = \lim_{t\to\infty} \frac{\sum_{v=0}^{t} a^v_{act(v)}}{t}. \tag{20}$$

□

4.3 Comparison: Adoption Ratio versus Usage History

We shall now compare the two different regimes from Sections 4.1 and 4.2.

Definition 7. *For two fixed points, x^*_α, x^*_β, say that x^*_α is* more mixed *(*less mixed*) than x^*_β if $|x^*_\alpha - 0.5| < |x^*_\beta - 0.5|$ ($|x^*_\alpha - 0.5| > |x^*_\beta - 0.5|$).*

Claim 8. *The two dynamic processes* Adoption Ratio *and* Usage History *exhibit, in general, different behavior and long-run outcomes. This holds true for both the unperturbed and perturbed dynamics.*

In the perturbed dynamics, ceteris paribus, Usage History *favors more extreme outcomes than* Adoption Ratio*. That is, a more mixed fixed point is less stable than a less mixed fixpoint under* Usage History*. Under* Adoption Ratio *the stability does not depend on whether a state is more or less mixed.*

In order to understand the claim, first consider the unperturbed processes. By Theorems 3 and 5 we have that the set of fixed points only coincides if and only if 0 and/or 1 are fixed points (and there are no other fixed points) or the best reply of any player is independent of social influence.

Next consider the perturbed dynamics. If the unperturbed dynamic has an absorbing state for *Usage History* the stable states of the two dynamics coincide in the case of a unique fixed point of *Agg*. Otherwise fixed points may differ (see Theorems 4 and 6).

Finally, for the case where the dynamic of *Usage History* has no absorbing states and turning to the second part of the theorem, note that under the perturbed *Adoption Ratio* the shift from one fixed point to another, say x^*_α to $x^*_{\alpha+1}$, is governed by the erroneous behavior of players currently *not* playing the innovation (action 1). In contrast, under the perturbed *Usage History* the shift from one fixed point to another, say x^*_α to $x^*_{\alpha+1}$, is governed by higher frequency of choices. Then, under *Adoption Ratio* the stability of a fixed point is independent of whether it is more or less mixed than another fixed point. Under perturbed *Usage History* the stability of a fixed point is higher for less mixed states, when all else is equal. This is the case since it is less likely for a very small number of users to use a product often enough to 'skew' the observation compared to a more mixed state.

Appendix C elaborates on the characteristic footprints in the distribution of s^{AR} and s^{UH} for the perturbed dynamics. It turns out that s^{AR} is not binomially distributed but s^{UH} is. In addition the variance of the two processes are, in general, different. This enables us to empirically discriminate which process is at work and thus to inform policy interventions or marketing campaigns.

5 Example

We shall now formalize our example introduced in the introduction. Consider bicycle usage and suppose that some commuters use the bicycle irrespective of its popularity, say innovators. Further assume that there is an early and late majority who may use the bicycle if enough others use it. Finally there are some non-adopters who will never use a bicycle for their daily commute. In particular assume the following population shares and thresholds:

- 5% *innovators*, always us the bicycle for their commute to work (action 1), that is, they play the innovation independent of social influence and hence their tipping value is 'negative' ($\mu_{\text{innovators}} < 0$),
- 45% *early majority*, who use the bicycle if at least 'few' use it (e.g., $\mu_{\text{early majority}} = 25\%$),
- 40% *late majority*, who use the bicycle if at least 'many' use it (e.g., $\mu_{\text{late majority}} = 75\%$),
- 10% *non-adopters*, who never play the innovation ($\mu_{\text{non-adopters}} > 1$).

Figure 1 shows the function *Agg* and the fixed points x_1^*, x_2^*, x_3^*. We invite the reader to verify that the fixed points of *Agg* are $x_1^* = 5\%$, $x_2^* = 50\%$, $x_3^* = 90\%$.

We compute the long-run stable state under *Adoption Ratio* according to Theorem 4. One finds the following resistances[10]

$$r_{x_1^*,x_2^*} = 0.2,\ r_{x_2^*,x_3^*} = 0.25,\ r_{x_2^*,x_1^*} = 0.25,\ r_{x_3^*,x_2^*} = 0.15$$

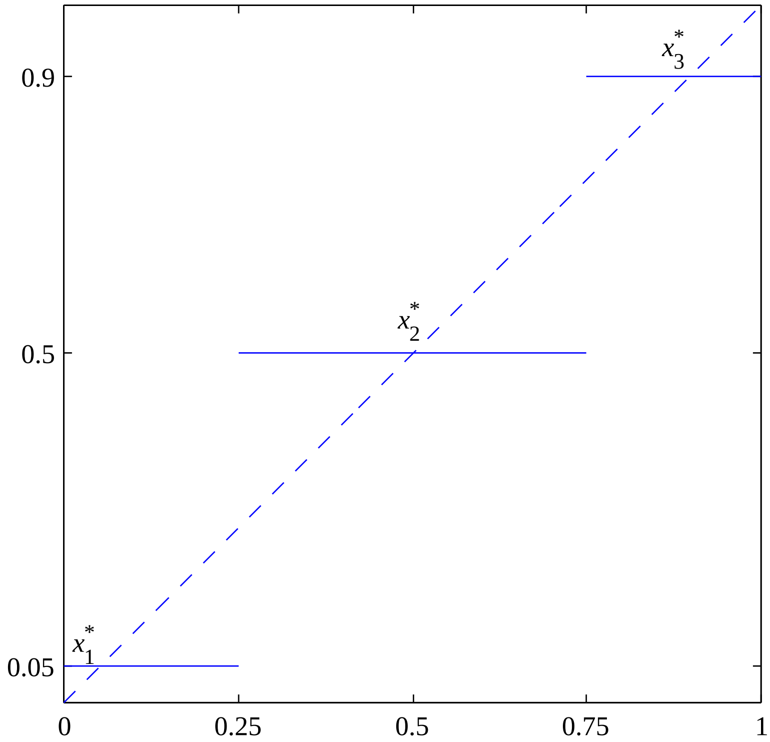
and the stochastic potentials

$$\gamma_{x_1^*} = 0.4,\ \gamma_{x_2^*} = 0.35,\ \gamma_{x_3^*} = 0.45.$$

Thus the (unique) stochastically stable state is x_2^* since it uniquely minimizes stochastic potential.

[10]Note that to be precise we need to define the population size and add resistance one to each of the formulas below. But this does not change the result.

Figure 1: *Agg*: x-axis shows $\mu(i)$, y-axis shows *Agg*



Next, we compute the long-run stable state under *Usage History* according to Theorem 6. The rounded results of Equation (16) are:

$$x_1^* : \ 1.05, \ x_2^* : \ 0.96, \ x_3^* : \ 0.92.$$

Thus the long-run observed frequency of action 1 is given by x_1^*.

This further shows, by example, that the two different processes *Adoption Ratio* and *Usage History* may yield significantly different outcomes. Note that the example is generic in the following sense: We can define "close-by" distributions with the same outcome.

To further build intuition for Claim 8 consider fixed point $x_1^* = 5\%$. In order to reach the basin of attraction of x_2^* the time series of choices (over the last k periods) needs to be at least 25%. That is, in the last k periods, players for whom 1 is currently the best response (5% of the population) need to be activated at least $25\% \cdot k$ times. That is, on average such a player needs to be activated at least 5 times as often as players whose best response is currently 0. On the other hand, suppose we are currently in $x_2^* = 50\%$. In order to reach the basin of attraction of x_1^* the time series of choices (over the last k periods) needs to be at most 25%. That is, in the last k periods, players for whom 0 is currently the best response (50% of the population) need to be activated at least $(100\% - 25\%) \cdot k$ times. That is, on average such a player needs to be activated at least 1.5 times as often as players whose best response is currently 1.[11] Since 1.5 is less than 5 it follows that the latter transition is more likely than the former.

[11]This follows from the simple calculation $(100\% - 25\%)/50\% = 1.5$.

6 Conclusion

In this paper we have studied the dynamics of social influence. We considered two different processes of social influence. On the one hand, social influence arises from the *Adoption Ratio* of the agents' actions. On the other hand, social influence arises from the *Usage History* of choices. We first identified the equilibria of the two processes and studied their stability in stochastic environments. We then showed that the outcomes may be very different. The reason being that one process is driven by fluctuations of non-adopters while the other process is driven by fluctuations of adopters. In particular, ceteris paribus, *Usage History* leads to more extreme outcomes than *Adoption Ratio*. Thus one needs to carefully examine the specific process of social influence at hand in order to be able to predict outcomes and design interventions. Returning to our example on bicycle usage discussed in the introduction the knowledge of which process is at hand may inform whether an intervention to promote the purchase of a bicycle (e.g., Cyclescheme) or an intervention to promote the usage of bicycles (e.g., Boris Bikes) is more apt.

References

Asch, S. E. (1955), "Opinions and social pressure." *Scientific American*, 193, 31–35.

Babichenko, Y. (2013), "Best-reply dynamics in large binary-choice anonymous games." *Games and Economic Behavior*, 81, 130–144.

Baddeley, M. (2010), "Herding, social influence and economic decision-making: Socio-psychological and neuroscientific analyses." *Phil. Trans. R. Soc. B*, 365, 281–290.

Banerjee, A. V. (1992), "A simple model of herd behavior." *Quarterly Journal of Economics*, 107, 797–817.

Bass, F. M. (1969), "A new product growth model for consumer durables." *Management Science*, 15, 215–227.

Bikhchandani, S., D. Hirshleifer, and I. Welch (1992), "A theory of fads, fashion, custom, and cultural change as informational cascades." *Journal of Political Economy*, 100, 992–1026.

Bongaarts, J. and S. C. Watkins (1996), "Social interactions and contemporary fertility transitions." *Population and Development Review*, 22, 639–682.

Bouchaud, J.-P. (2013), "Crises and collective socio-economic phenomena. cartoon models and challenges." *Journal of Statistical Physics*, 151, 567–606.

Bramoulle, Y. (2007), "Anti-coordination and social interactions." *Games and Economic Behavior*, 58, 30–49.

Butler, K. and M. Stephens (1993), "The distribution of a sum of binomial random variables." Technical Report 467, Stanford University, April 1993, prepared for the Office of Naval Research.

Cabinet Office (2012), *Applying behavioural insights to reduce fraud, error and debt.* Behavioural Insights Team UK.

Cukierman, A. (1991), "Asymmetric information and the electoral momentum of public opinion polls." *Public Choice*, 70, 181–213.

de Silva, H., D. Ratnadiwakara, and A. Zainudeen (2011), "Social influence in mobile phone adoption: Evidence from the bottom of the pyramid in emerging asia." *Information Technologies & International Development*, 7, 1–18.

Devenow, A. and I. Welch (1996), "Rational herding in financial economics." *European Economic Review*, 40, 603–615.

Dodds, P. S. and D. J. Watts (2005), "A generalized model of social and biological contagion." *Journal of Theoretical Biology*, 232, 587–604.

Dohmen, T., A. Falk, D. Huffman, and U. Sunde (2012), "The intergenerational transmission of risk and trust attitudes." *The Review of Economic Studies*, 79, 645–677.

Easley, D. and J. Kleinberg (2010), *Networks, crowds, and markets: Reasoning about a highly connected world.* Cambridge University Press.

Fehr, E. and K. Hoff (2011), "Introduction: Tastes, castes and culture: the influence of society on preferences." *The Economic Journal*, 121, 396–412.

Foster, D. and H. P. Young (1990), "Stochastic evolutionary game dynamics." *Theoretical Population Biology*, 38, 219–232.

Goetzke, F. and T. Rave (2011), "Bicycle use in Germany: Explaining differences between municipalities with social network effects." *Urban Studies*, 48, 427–437.

Granovetter, M. (1978), "Threshold models of collective behavior." *The American Journal of Sociology*, 83, 1420–1443.

Hamilton, W. D. (1971), "Geometry for the selfish herd." *Journal of Theoretical Biology*, 31, 295–311.

Horst, U. and J. A. Scheinkman (2006), "Equilibria in systems of social interactions." *Journal of Economic Theory*, 130, 44–77.

Kandori, M., G. J. Mailath, and R. Rob (1993), "Learning, mutation, and long run equilibria in games." *Econometrica*, 61, 29–56.

Keynes, J. M. (1936), *The General Theory of Employment, Interest and Money.* Macmillan.

Le Bon, G. (1895), *Psychologie des foules.*

Loeper, A., J. Steiner, and C. Stewart (2014), "Influential opinion leaders." *The Economic Journal*, 124, 1147–1167.

Lopez-Pintado, D. and D. J. Watts (2008), “Social influence, binary decisions and collective dynamics.” *Rationality and Society*, 20, 399–443.

Macy, M. W. (1991), “Chains of cooperation: Threshold effects in collective action.” *American Sociological Review*, 56, 730–747.

Maes, M. and K.-D. Opp (2016), “When is ignorance bliss? Disclosing true information and cascades of norm violation in networks.” *Social Networks*, 47, 116–129.

Meade, N. and T. Islam (2006), “Modelling and forecasting the diffusion of innovation - a 25-year review.” *International Journal of Forecasting*, 22, 519–545.

Neary, P. R. (2012), “Competing conventions.” *Games and Economic Behavior*, 76, 301–328.

Newton, J. and R. Sawa (2015), “A one-shot deviation principle for stability in matching problems.” *Journal of Economic Theory*, 157, 1–27.

Peski, M. (2010), “Generalized risk-dominance and asymmetric dynamics.” *Journal of Economic Theory*, 145, 216–248.

Pinsky, R. G. (2013), “The speed of a random walk excited by its recent history.” Working Paper, arXiv: 1305.7242.

Rogers, E. M. (1962), *Diffusion of Innovations*. The Free Press.

Salganik, M. J., P. S. Dodds, and D. J. Watts (2006), “Experimental study of inequality and unpredictability in an artificial cultural market.” *Science*, 311, 854–856.

Scharfstein, D. S. and J. C. Stein (1990), “Herd behavior and investment.” *American Economic Review*, 80, 465–479.

Schelling, T. C. (1971), “Dynamic models of segregation.” *Journal of Mathematical Sociology*, 1, 143–186.

Schelling, T. C. (1978), *Micromotives and Macrobehavior*. Norton & Company.

Schultz, P. W., J. M. Nolan, R. B. Cialdini, N. J. Goldstein, and V. Griskevicius (2007), “The constructive, destructive and reconstructive power of social norms.” *Psychological Science*, 18, 429–434.

Shiller, R. J. (2000), *Irrational exuberance*. Princeton University Press.

Teng, J. T. C., V. Grover, and W. Guettler (2002), “Information technology innovations: General diffusion patterns and its relationships to innovation characteristics.” *IEEE Transactions on Engineering Management*, 49, 13–27.

Trotter, W. (1916), *Instincts of the Herd in War and Peace*.

Young, H. P. (1993), “The evolution of conventions.” *Econometrica*, 61, 57–84.

Young, H. P. (2009), "Innovation diffusion in heterogeneous populations: contagion, social influence, and social learning." *American Economic Review*, 99, 1899–1924.

Young, H. P. (2011), "The dynamics of social innovation." *PNAS*, 108, 21285–21291.

A Proof of Lemma 2

Proof of Lemma 2. We shall first show that there exists a fixed point. If $Agg(1) = 1$ then 1 is a fixed point and similarly if $Agg(0) = 0$, 0 is a fixed point. Thus we still have to consider the case where $0 < Agg(1) < 1$ since Agg is increasing. Define the function:

$$g : [0,1] \rightarrow \left\{-1, ..., \frac{-1}{p}, 0, \frac{1}{p}, ..., 1\right\} \tag{21}$$

$$g(x) = Agg(x) - x \tag{22}$$

It suffices to prove that there exists x such that $g(x) = 0$. Since $Agg(0) > 0$ we have $g(0) \neq 0$ and thus $g(0) > 0$. Also $Agg(1) < 1$ and thus $g(1) < 0$. Hence there must eventually be a change from positive to negative sign with increasing x. Remember that g is defined on $\left\{-1, ..., \frac{-1}{p}, 0, \frac{1}{p}, ..., 1\right\}$. We have by monotonicity of Agg that for every $x < 1$, $g\left(x + \frac{1}{p}\right) \geq g(x) - \frac{1}{p}$, that is, the function g decreases at most in steps of $\frac{1}{p}$. Therefore there exists x^* such that $g(x^*) = 0$ and thus $f(x^*) = x^*$.

For the second part of the lemma note that if x^* is a fixed point of Agg all players of the same type necessarily play the same action. It remains to show that for any fixed point $\bar{q}_k$ we have $\mu_k < \bar{q}_k < \mu_{k+1}$. By contradiction, suppose there exists a fixed point $\bar{q}_k < \mu_k$. Then for the share of players q_j with $j \geq k$ it is optimal to play m when observing $\bar{q}_k$, given that their tipping value is greater or equal than μ_k and thus $\bar{q}_k$ is not a fixed point. A similar argument applies for $\bar{q}_k > \mu_{k+1}$. □

B Proof of Theorems 3 and 4

Proof of Theorem 3. By contradiction, suppose there exists an absorbing state $\mathbf{a}^*$ with $x^* \in [0,1]$ such that x^* is not a fixed point of Agg. Suppose $x^* > Agg(x^*)$ (the other case is analogous. That is, the number of players playing 1 is x^*, however the number of players for whom it is desirable to play 1 is $Agg(x^*)$ which is strictly smaller than x^*. Hence there exists at least one player i playing 1 for whom $\mu(i) > x^*$ and if i is activated in a given period he will change his action from 1 to 0. Thus the state $\mathbf{a}^*$ is not an absorbing state. Since by Lemma 2 Agg has at least one fixed point, for a given fixed point x^* we can assign an action to each player such that the action profile is an absorbing state. We simply order the players by their tipping values (from small to large) and assign action 1 successively to players until the proportion x^* is reached. Finally note that the dynamic is not ergodic if multiple absorbing states exist and therefore the absorbing states attainable depend on the initial state $\mathbf{a}^0$. □

In order to prove Theorem 4 we shall first introduce stochastic stability analysis:

B.1 Stochastic stability

Consider the stochastic process governing the change of $\bar{a}$. The process is Markovian and its recurrent classes are characterized by the fixed points of Agg. It is a regular perturbed Markov process and we can therefore use stochastic stability analysis (Foster and Young 1990, Kandori et al. 1993, Young 1993). We consider the long-run behavior of the process when ε becomes small. Note that the perturbed process is ergodic for $\varepsilon > 0$ and thus has a unique stationary distribution, say Π_ε over the state space $[0, \frac{1}{p}, \ldots, 1]$. We thus study $\lim_{\varepsilon \to 0} \Pi_\varepsilon = \Pi_0$.

Definition 9. *A state $\bar{a} \in [0, \frac{1}{p}, \ldots, 1]$ is* stochastically stable *if $\Pi_0(\bar{a}) > 0$. Denote the set of stochastically stable states by* $\mathbf{S}$.

For a given parameter ε denote the probability of transiting from $\bar{a}$ to $\bar{a}'$ in one period by $\mathbb{P}_\varepsilon[\bar{a}, \bar{a}']$. The *resistance* of a transition $\bar{a} \to \bar{a}'$ is the unique real number $r(\bar{a}, \bar{a}') \geq 0$ such that $0 < \lim_{\varepsilon \to 0} \mathbb{P}_\varepsilon[\bar{a}, \bar{a}']/\varepsilon^{r(\bar{a}, \bar{a}')} < \infty$. For completeness let $r(\bar{a}, \bar{a}') = \infty$ if $\mathbb{P}_\varepsilon[\bar{a}, \bar{a}'] = 0$. Hence a transition with resistance r has probability of the order $O(\varepsilon^r)$. We shall call a transition (possibly in multiple periods) $\bar{a} \to \bar{a}'$ a *least cost transition* if it exhibits the lowest order of resistance. That is, let $\bar{a}, \bar{a}_1, \ldots, \bar{a}_k = \bar{a}'$ (k finite) be a path of one-period transitions from $\bar{a}$ to $\bar{a}'$. Then a least-cost transition minimizes $\sum_{l=0}^{k-1} r(\bar{a}_l, \bar{a}_{l+1})$ over all such paths. By abuse of notation we write $r(\bar{a}, \bar{a}') = \sum_{l=0}^{k-1} r(\bar{a}_l, \bar{a}_{l+1})$.

Young (1993) shows that the computation of the stochastically stable states can be reduced to an analysis of rooted trees on the set of recurrent classes of the unperturbed dynamic. Identify the recurrent classes with the nodes of a graph. Given a node $\bar{a}$, a collection of directed edges T forms an *$\bar{a}$-tree* if from every node $\bar{a}' \neq \bar{a}$ there exists a unique outgoing edge in T, the graph has no cycles.

Definition 10. *The resistance $r(T)$ of a $\bar{a}$-tree T is the sum of the resistances of its edges. The* stochastic potential *of $\bar{a}$, $\gamma(\bar{a})$, is given by*

$$\gamma(\bar{a}) = \min\{r(T) : T \text{ is an } \bar{a}\text{-tree}\}. \tag{23}$$

Theorem 4 in Young (1993) states that the stochastically stable states are precisely those states where γ is minimized.

Proof of Theorem 4. Let $x_1^*, ..., x_l^*$ (in increasing order) be the fixed points of Agg. In order to pass from one to another one needs to pass through all the fixed points in between. The process governing the change of $\bar{a}$ has a linear transition structure, namely, to go from state $\frac{\alpha}{p}$ to $\frac{\beta}{p}$ one has to pass through all the states $\frac{\alpha+1}{p}, \ldots, \frac{\beta-1}{p}$ for $\alpha < \beta$ and similarly through all the states $\frac{\alpha-1}{p}, \ldots, \frac{\beta+1}{p}$ for $\alpha > \beta$. To find the least resistant paths between two recurrent classes it suffices to calculate the resistance between *neighboring* recurrent classes.

We shall give the argument for $r_{x^*_\alpha, x^*_{\alpha+1}}$, the one for $r_{x^*_{\alpha+1}, x^*_\alpha}$ is analogous. We define several groups of players. Let P_1^+ be the set of players who played 1 in the last period and for whom 1 is currently a best reply and let P_1^- be the set of players who played 1 in the last period and for whom 0 is currently the best reply. Define P_0^+ and P_0^- analogously. Since we consider the resistance between two absorbing states we have for the starting state that $P_1^- = P_0^- = 0$. Given that $x^*_\alpha < x^*_{\alpha+1}$ we have that in $x^*_{\alpha+1}$ more players play 1. Suppose that we construct a path such that only players who currently play 0 switch to 1. Then any such switch is erroneous behavior as long as $x > Agg(x)$. Once $x < Agg(x)$ further transitions have resistance zero. For any given x with $x > Agg(x)$ one needs at least $(x - Agg(x)) \cdot p + 1$ errors to enter a region where $x^*_{\alpha+1}$ becomes an attractor. (Note that $x - Agg(x) = x - x^*_\alpha$ for $x^*_\alpha \leq x < x^*_{\alpha+1}$.) Since this must hold for all x with $x > Agg(x)$ we have

$$r_{x^*_\alpha, x^*_{\alpha+1}} = \max_{x \in [x^*_\alpha, x^*_{\alpha+1}]} \{x - Agg(x)\} \cdot p + 1. \tag{24}$$

This is sufficient since after this many trembles there exists a zero resistance path moving to the neighboring absorbing state associated with $x^*_{\alpha+1}$. This proves the claim.

We can now conclude that the stochastic potential of a fixed point is given by

$$\gamma_{x^*_\alpha} = \sum_{\beta=1}^{\alpha-1} r_{x_\beta, x_{\beta+1}} + \sum_{\beta=\alpha+1}^{l} r_{x_\beta, x_{\beta-1}}. \tag{25}$$

The first summand gives the resistances of passing from the rightmost fixed point to x^*_α, the second the resistance from passing from the leftmost fixed point to x^*_α. Now by Young (1993) we have that the stochastically stable states are precisely those states which have minimal stochastic potential. □

C Empirically discriminating between *Adoption Ratio* and *Usage History*

We here show characteristic footprints in the distribution of s^{AR} and s^{UH} for the perturbed dynamics that can be used to empirically discriminate which process is at work (as long as $x^* = 0.5$ is currently not the observed fixed point and s^{UH} has no absorbing states in the unperturbed process). We analyze the behavior of s^{AR} and s^{UH} around an interior fixed point x^* of the dynamic. We shall study the distribution of s^{AR} and s^{UH} conditional on remaining in the basin of attraction of x^*.

Distribution of s^{AR}. Suppose that ε is fixed to a value greater than zero. Note that within a basin of attraction a player's best response remains constant. Then a player's action, when activated to revise, is picked independently of the other players' actions. Given his response function f_i, he plays his best response with probability $1 - \frac{\varepsilon}{2}$ and the other action with probability $\frac{\varepsilon}{2}$ (Bernoulli trial). That is, $p \cdot x^*$ players play d with probability $f_i = 1 - \varepsilon/2$ and m otherwise, and $p \cdot (1 - x^*)$ players play d with probability

$f_i = \varepsilon/2$ and m otherwise. We thus have for s^{AR}:

$$s^{AR} \sim \frac{1}{p}\left(\sum_{i\in P: f_i = 1-\varepsilon/2} B(1, 1-\frac{\varepsilon}{2}) + \sum_{i\in P: f_i=\varepsilon/2} B(1,\frac{\varepsilon}{2})\right) \tag{26}$$

$$\sim \frac{1}{p}\Big(p\cdot x^*\cdot B(1,1-\frac{\varepsilon}{2}) + p\cdot(1-x^*)\cdot B(1,\frac{\varepsilon}{2})\Big) \tag{27}$$

$$\sim \frac{1}{p}\Big(B(p\cdot x^*, 1-\frac{\varepsilon}{2}) + B(p\cdot(1-x^*),\frac{\varepsilon}{2})\Big) \tag{28}$$

where Equation (28) holds since the sum of iid binomials is again binomial with the same parameter. As shown by Butler and Stephens (1993) there is no closed form for the distribution of a sum of binomials with different parameters, but they derive a recursive formula. In particular for the reduced form in Equation (26) of two binomials one easily finds for the mean of s^{AR}:

$$\mathbb{E}(s^{AR}) = x^*\cdot(1-\frac{\varepsilon}{2}) + (1-x^*)\cdot\frac{\varepsilon}{2} \tag{29}$$

$$= x^* + (1-2x^*)\cdot\frac{\varepsilon}{2} \tag{30}$$

and for the variance of s^{AR}:

$$Var(s^{AR}) = x^*\cdot(1-\frac{\varepsilon}{2})\cdot\frac{\varepsilon}{2} + (1-x^*)\cdot\frac{\varepsilon}{2}\cdot(1-\frac{\varepsilon}{2}) \tag{31}$$

$$= \frac{\varepsilon}{2}\cdot(1-\frac{\varepsilon}{2}) \tag{32}$$

Distribution of s^{UH}. As before, within a basin of attraction a player's best response remains constant. Since we consider the case where the unperturbed process has no absorbing states we can assume that $\varepsilon = 0$. Then the next action (by the player activated in the next period) is given by a binomial distribution with parameter x^* (since players best respond with probability 1). The intuition, that s^{UH} follows the sum of k binomial distributions with parameter x^* is wrong, since the order of occurrence matters and thus $s^{UH}(t)$ is correlated with $s^{UH}(t+i)$ for all $i = 1,\ldots,k-1$. By considering non-overlapping windows of length k we can recover independence and thus have that for $j = 1,2,3,\ldots$ $s^{UH}(k\cdot j)$ are independent Bernoulli trials with parameter x^*. Thus $s^{UH}(k\cdot t)$ is binomial distributed with parameter x^* and one finds for the mean of s^{UH}:

$$\mathbb{E}[s^{UH}(k\cdot t)] = x^* \tag{33}$$

and for the variance of s^{UH}:

$$Var(s^{UH}(k\cdot t)) = x^*\cdot(1-x^*) \tag{34}$$

To summarize s^{AR} is not binomially distributed (for $x^* \neq 0.5$) and s^{UH} is binomially distributed. Further the variances of the two processes are, in general, different. Thus we can employ standard statistical tests to identify which process, *Adoption Ratio* or *Usage History*, is underlying a given sample of the aggregate observation data. Note that it is not necessary to have any additional information, in particular it is not necessary to know the players' thresholds or order of activation. The first test we can use is whether the observed data is binomially distributed. If this yields a statistically significant result we are done. But if ε is very small it may be that the result of this test is not statistically significant. Then a second test can be used. The variance of *Usage History* is dependent on the fixed point (that is the mean) and we can therefore use this fact to discriminate between the two processes. In particular, if the fixed point is not too close to $all - m$ or $all - d$ we can use this test. Finally, if data of the behavior around two fixed points is available a third test can be performed. For *Adoption Ratio* the variance is the same around all fixed points whereas the variance for *Usage History* differs around each fixed point.