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Quick or cheap?

2

Breaking points in dynamic markets*

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4

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5

Abstract

6

7 We examine two-sided markets where players arrive stochastically over
 8 time and are drawn from a continuum of types. The cost of matching a
 9 client and provider varies, so a social planner is faced with two contending
 10 objectives: *a)* to reduce players' *waiting time* before getting matched; and
 11 *b)* to form efficient pairs in order to reduce *matching costs*. We show that
 12 such markets are characterized by a *quick or cheap* dilemma: Under a large
 13 class of distributional assumptions, there is no 'free lunch', i.e., there exists
 14 no clearing schedule that is simultaneously optimal along both objectives.
 15 We further identify a unique breaking point signifying a stark reduction in
 16 matching cost contrasted by an increase in waiting time. Generalizing this
 17 model, we identify two regimes: one, where no free lunch exists; the other,
 18 where a window of opportunity opens to achieve a free lunch. Remarkably,
 greedy scheduling is never optimal in this setting.

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1 Introduction

Many economic interactions require the dynamic matching of heterogeneous agents that arrive stochastically to a two-sided market. Examples include the dynamic matching of clients and providers in markets for jobs and services, of buyers and sellers in financial markets, of taxis and passengers on road networks, of donors and recipients in organ exchanges, etc.¹

It is known that many of these markets vary substantially in terms of efficiency (Roth and Xing, 1994, 1997). The focus of our investigation is on a crucial aspect of market design in this context, namely the *scheduling of clearing events*. The goal of “making a thick market” (Roth, 2012) is to find the best schedule of market clearing so that sufficient clients and providers are in the market to allow for stable and efficient matches over time while not waiting excessively. Designing an optimal clearing policy thus requires optimizing along the following two objectives:

1. To reduce the coexistence of agents on the two sides of the market.
2. To match parties in such a way so as to minimize cost (or maximize productivity).

In pursuit of these two goals, clearing schedules need to be formulated to address the following key question: *How long should the social planner wait between two clearing events?*

To illustrate the above, consider the example of a governmental employment bureau faced with a dynamically evolving job market where job offers and job seekers arrive to the system stochastically over time. The bureau has two aims, namely to reduce the coexistence of vacancies and job seekers, and to match vacancies with the skills of individual job seekers so as to maximize productivity. Waiting times incur costs via unemployment benefits, as well as costs due to productivity losses incurred by badly staffed vacancies.

To gain in generality, we abstract away from application-specific details (such as the particular structure of the application and recruitment processes). This allows us to focus on the trade-offs between two different and concurrent objectives, waiting time and matching cost. Perhaps surprisingly, this *quick or cheap* dilemma is not easily resolvable as greedy scheduling policies are generally not optimal in this context.

¹More generally, we focus on markets for ‘nondurables’; for classic studies on product durability and market performance see Smith (1962), Smith et al. (1988), and Dickhaut et al. (2012).

1 Related work

2 Dating back to the 1950s, the first related strand of work focuses on behavioral
3 aspects underlying the dynamics of unemployment and job vacancies in labor
4 markets (Dow and Dicks-Mireaux, 1958). These analyses identify avenues to re-
5 duce *waiting* – i.e., the coexistence of unemployment and vacancies – by better
6 understanding the behavior of job seekers and job providers. Lines of reason-
7 ing proposed to explain the coexistence of unemployment and vacancies include
8 the classical search models of McCall (1970), Mortensen (1970), and Lucas and
9 Prescott (1974), as well as more recent models with workforce inertia due to Shimer
10 (2007).² We complement this literature with a view that some degree of waiting
11 is actually beneficial from a social welfare perspective as it enables market thick-
12 ening – which in turn enables mismatch reduction. To illustrate this, consider the
13 example of Shimer (2007), where some laid-off steel workers are not immediately
14 given vacant positions as nurses. This may indeed be deemed optimal by a social
15 planner when – by delaying their match – these nurse vacancies eventually are
16 taken up by better nurses and the jobless steel workers find other jobs in the steel
17 industry that might become available in the future.³

18 The second strand of related work comes from the matching literature and extends
19 the canonical static matching framework to a dynamic setting.⁴ As in the exam-
20 ple of steel workers and nurses above, *mismatch* in dynamic environments may
21 occur due to temporal inconsistencies, whereby, a posteriori, better matches were
22 precluded by inferior matches that were formed earlier on. Therefore, some delay
23 may be optimal from a social planner perspective in order to reduce mismatch.
24 From a practical viewpoint, the challenge is to identify optimal mechanisms that
25 thicken and clear the market in a way that balances these two objectives.

26 In this regard, Akbarpour et al. (2017), Ashlagi et al. (2019), Baccara et al. (2018),
27 and Loertscher et al. (2018) break new ground in identifying optimal clearing
28 schedules.⁵ More precisely, Akbarpour et al. (2017), in the spirit of an organ ex-

²Note that Shimer (2007) terms his explanandum “mismatch” (as opposed to “waiting”), a term the matching literature uses to describe suboptimal matchings, which may be confusing.

³*Waiting* is explained behaviorally through inertia in Shimer (2007), that is, by the argument that steel workers stay close to their factories hoping that they reopen; Lucas and Prescott (1974) propose a different interpretation whereby waiting is due to the fact that steel workers must actively spend some time searching for these nursing jobs elsewhere.

⁴The canonical static frameworks underlying our analyses were pioneered by Egervary (1931); Koenig (1931), and Edmonds (1965); see also Gale and Shapley (1962) for matching with ordinal preferences.

⁵These are inspired by some earlier papers on dynamic matching in organ exchange by Uenver (2010); Zenios (2002). See Akbarpour et al. (2017) for a discussion. See also Bloch and Houy

1 change application such as the ‘kidney exchange’, identify the optimal mechanism
2 to maximize the number of matches, that is, to minimize the number of agents
3 perishing resulting from failing to get recipients matched with donors in time. In
4 the model of Akbarpour et al. (2017), agents from both sides of the market arrive
5 and leave stochastically and all carry identical match values, i.e., they are of the
6 same type (in the spirit of each life being worth the same). However, there are
7 two types of agents, as some matches are feasible and others are infeasible, thus
8 rendering some agents easier – others harder – to match.⁶ The optimal mechanism
9 identified by Akbarpour et al. (2017) minimizes the number of unmatched patients
10 based on information concerning arrivals and departures, which may involve delay-
11 ing compatible matches. Without such information, greedy scheduling is always
12 optimal in this setting. In fact, Ashlagi et al. (2019) show that greedy policies are
13 generally optimal, even if information about departure times is available when the
14 above kind of ‘kidney exchange’ markets becomes large.

15 In a related setting, Baccara et al. (2018) and Loertscher et al. (2018) introduce
16 binary diversifications of agents and the notion of waiting costs instead of perishing
17 rates as in Akbarpour et al. (2017). In Baccara et al. (2018), agents arrive in
18 donor-recipient pairs and recipients are allowed to decline matches in order to
19 remain in the market. This is motivated by the applications under scrutiny which
20 include, among others, child adoption. As a result, one of the study’s key focuses
21 is on strategic incentives and their role in determining market outcomes. Their
22 optimal clearing policy is discriminatory, in that it involves matching same-type
23 pairs greedily, and delaying up to some threshold when there are only cross-type
24 pairs in the market. By contrast, Loertscher et al. (2018) introduce a common
25 discount factor (instead of a constant waiting cost) for both types of agents (as
26 well as for the social planner).⁷ They focus on the analysis of the possibility of
27 efficient trade and rent extraction by the market maker.

28 In theoretical computer science, the study of related questions dates back at least
29 to the pioneering paper of Karp et al. (1990).⁸ To the best of our knowledge,

(2012), Kurino (2014), and Leshno (2012) who study related queuing models where one side of the market is already present (such as in the housing market).

⁶This can be modeled by means of a dynamically changing compatibility graph where edges represent feasible matches.

⁷The discount factor is motivated by the study’s focus on financial markets, and would be determined by risk-free rate, beta, and risk premium. Related analyses of financial markets include Budish et al. (2015) and Wah et al. (2015) (see also Wah et al. (2017) for market-making more generally) who consider periodic clearing of order books to abate certain market phenomena such as volatility that stem from high-frequency trading.

⁸Karp et al. (1990) and subsequent work – similar to its economic counterparts – focus on

1 Emek et al. (2016) were the first in this strand of research to consider the scenario
2 where all agents arrive on the market over time (instead of just one market side).
3 They present a non-bipartite model where requests arrive stochastically from one
4 of n different locations to study the performance of different algorithms in terms
5 of worst-case matching and waiting cost.⁹ In the setting of Emek et al. (2016), the
6 specific match costs result from the distance between agents' locations so that, for
7 a patient social planner, it is optimal to wait and only match agents who are at
8 the same location.

9 Finally, motivated by ride-sharing applications, Ashlagi et al. (2017) extend the
10 model of Emek et al. (2016) to a bipartite setting where agents independently
11 arrive at different locations. Ashlagi et al. (2017) study the performance of a
12 family of clearing schedules along the two axes of waiting vs. mismatch separately,
13 an approach that we extend in order to formulate the induced trade-off between
14 waiting time and the cost of matching.

15 Contributions of the paper

16 Our paper examines dynamic markets with an infinite type space (in contrast to
17 one or two types), a framework we call the *dynamic clearing game*. Our point of
18 departure is the static assignment game of Shapley and Shubik (1972) to which we
19 add a dynamic layer whereby clients and providers arrive to the market stochasti-
20 cally and independently. Skills are drawn from a large class of type distributions
21 so that every match is possible but some matches are more costly than others. At
22 each matching event, the social planner must decide who to match with whom,
23 and how long to wait before the next matching event. As such, the social planner
24 is called to weigh, on the one hand, mismatches incurred from matching clients
25 and providers suboptimally; and, on the other hand, the agents' waiting time. To
26 address this dual issue, we study clearing schedules in terms of *when* to match in a
27 fully heterogeneous setting where the social planner has no information regarding
28 the cost of matching couples currently in the market and has no information about
29 the future arrivals of individuals.

models with two market sides, where by contrast one side is typically present to begin with and incoming agents from the other side can only match with some of the present agents according to a compatibility graph (see Mehta (2013) for an overview and Aggarwal et al. (2011) for extensions to vertex-weighted matching).

⁹Azar et al. (2017) obtain additional results in terms of upper and lower bounds for the original model. Emek et al. (2019) obtain sharper results for a two-location model. There are also other extensions such as allowing for a stochastic graph (Anderson et al., 2015; Ashlagi et al., 2018).

1 For concreteness, we start by studying a micro-level model where costs of in-
2 dividual matches are distributed according to independent exponential random
3 variables. Whilst our results hold for other distributions too, this model has the
4 advantage of being tractable in closed form. In more detail, we first establish a
5 class of optimal clearing schedules for two extreme types of single-objective social
6 planners – that is, for social planners who only care about minimizing waiting
7 time (in which case greedy is best) or mismatch costs (resulting in endless delay),
8 but not both at the same time. Second, we show that these two objectives are
9 mutually incompatible, and multi-objective social planners (who care about both)
10 face a fundamental trade-off. Specifically, there is no ‘free lunch’, that is *there is*
11 *no clearing schedule that is approximately optimal in terms of both waiting time*
12 *and matching cost*. Remarkably, the greedy clearing schedule is sub-optimal for
13 every multi-objective social planner.

14 Building on the no free lunch result, we proceed to fill the spectrum between
15 matching cost and waiting time minimization. We do so by introducing a class
16 of clearing schedules covering a wide range of social planning desiderata between
17 waiting time and matching cost, and achieving a continuous trade-off between the
18 two. To explore the finer aspects of this trade-off, we introduce a utility model for
19 the social planner whereby the associated utility of matching cost is of the same
20 order as the agents’ utility of waiting time. Under this model, we show that there
21 exists a non-trivial clearing schedule achieving this balance, and we show that this
22 schedule is effectively unique (up to asymptotic order considerations).

23 Finally, we generalize our key findings by studying different decay rates of match-
24 ing costs (instead of focusing on one decay rate that results from the micro-founded
25 match costs). We identify two regimes. One, where no free lunch continues to hold.
26 The other, where the benefit from waiting is growing quickly enough, such that a
27 window of opportunity opens and it *is* possible to get a free lunch. As before, in
28 both regimes, greedy scheduling is generally sub-optimal.

29 Compared to the existing literature on the trade-off between waiting and mis-
30 match (both in economics and computer science) our model introduces *incomplete*
31 *information* about the distribution of past and future match costs and considers
32 an infinite type space in a tractable model. As a consequence, the social planner
33 tries to resolve the trade-off between matching optimally and waiting time in light
34 of incomplete information. Incomplete information in our setting implies that the
35 social planner must employ clearing schedules that do not take as input the rel-
36 ative strengths of current and future matches (since the latter is unknown), thus

1 yielding qualitatively new results.

2 In contrast to prior results for markets with one or two types of match costs where
3 lack of information resulted in optimality of some form of greedy scheduling, we
4 find that greedy clearing is generally not optimal in the presence of many types.
5 Hence, the quick-versus-cheap trade-off is more intricate than previously found.
6 Moreover, our results may actually also have consequences for applications that
7 have been studied before too (e.g. kidney exchange) if other match value metrics
8 (e.g. potential years of life lost or disability-adjusted life years) are used that
9 would produce more than binary match values. By studying fully heterogeneous
10 match costs we have to rely on different mathematical tools compared to previous
11 analyses, which were often able to reduce the induced dynamics to discrete Markov
12 processes.

13 The key technical innovations of our paper concern the concurrent consideration of
14 a continuum of types, independent arrivals, and incomplete information. In turn,
15 these contributions rely on a range of previously unused tools from probability
16 theory and disordered systems to obtain closed-form solutions. These underly-
17 ing results are concerned with the expected matching cost for given instances of
18 random, static assignment games. In particular, in static assignment games with
19 the same number of clients and providers and $\exp(1)$ distributed edge weights,
20 Aldous (2001) proved the long-standing conjecture that the expected minimum
21 weight matching converges to $\pi^2/6$ (i.e., as the number of players is growing).
22 This result was later extended by Waestlund (2005) to assignment games with
23 match costs drawn from non-identical exponential distributions.¹⁰ By leveraging
24 the techniques of Aldous (2001) and Waestlund (2005), we are able to compute
25 the expected matching cost for every ‘snapshot in time’ of the dynamic clearing
26 game. This provides strong foundations for our proofs which are then focused
27 on estimating the fluctuations that result from the random arrival of clients and
28 providers and their randomly drawn match costs. To achieve this, we use several
29 approximation techniques (in particular, the approximation of the arrival process
30 by a continuous-time Wiener process), which allow us to port over several results
31 from martingale limit theory (such as the law of the iterated logarithm).

¹⁰To the best of our knowledge, the work of Walkup (1979) is the first to pose the question, while Mezard and Parisi (1987) conjectured the specific limit value. We also leverage the analyses of Buck et al. (2002) and Linusson and Waestlund (2004) who obtain results for the expected values of finite instances of the latter models, showing – as a byproduct – that the value is increasing with the number of agents. For a survey of this literature, we refer the reader to Krokmal and Pardalos (2009).

1 **Paper outline.** The rest of the paper is structured as follows. In Section 2,
2 we introduce the *dynamic clearing game* and the performance measures relevant
3 for our analysis. In Section 3, we show that a social planner who cares about
4 both waiting and mismatch faces a fundamental and non-negligible trade-off. We
5 then go on to analyze a natural selection of clearing schedules in Section 4, which
6 cover the whole range of possible trade-offs. In Section 5, we commit to a specific
7 utility function that specifies how the social planner values waiting time versus
8 matching cost and find the unique optimal clearing schedule. Section 6 generalizes
9 the analysis and shows that there are two regimes, one where ‘free lunch’ is not
10 achievable and one where it is achievable. Finally, in Section 7, we discuss practical
11 implications and of avenues for future research.

12 2 The model

13 In this section, we introduce the model, which we shall refer to as the *dynamic*
14 *clearing game*.

15 **The dynamic clearing game.** Consider the following model of a dynamic two-
16 sided market evolving in continuous time $\tau \in [0, \infty)$. At each tick of a Poisson
17 clock with rate 1 an agent enters the market; this agent could be either a *client*
18 or a *provider*, with equal probability.¹¹ To keep track of the number of agents in
19 both sides of the market, let $\mathcal{C}(\tau)$ and $\mathcal{P}(\tau)$ denote the set of clients and providers
20 that have entered the market by time τ (and possibly already left again), and let
21 $N_{\mathcal{C}}(\tau) = |\mathcal{C}(\tau)|$ and $N_{\mathcal{P}}(\tau) = |\mathcal{P}(\tau)|$ be the respective numbers thereof. Then,
22 the number of agents on the *short side of the market* will be written $N(\tau) =$
23 $\min\{N_{\mathcal{C}}(\tau), N_{\mathcal{P}}(\tau)\}$.¹²

24 As in the static assignment model of Shapley and Shubik (1972) on which we build,
25 we consider a one-to-one matching market where each client is to be *matched* to
26 at most one provider and vice versa; then, once a couple is matched, both agents
27 leave the market. For example, in labor market language, each job seeker gets

¹¹We are using here the terms ‘client’ and ‘provider’ in a generic sense, just to illustrate the difference between the two sides of the market. As we explain below, what is important from a modeling perspective is that ‘clients’ are to be matched to ‘providers’ (as in our running example of job vacancies and job seekers).

¹²In a slight (but convenient) abuse of notation, we will sometimes write $N_{\mathcal{C}}(t)$, $N_{\mathcal{P}}(t)$, and $N(t)$ to denote respectively the number of clients, providers, and agents at the short side of the market when the t -th agent enters the market – specifically, letting $\tau(t)$ denote the time at which the t -th agent enters the market, we will write $N_{\mathcal{C}}(t) \equiv N_{\mathcal{C}}(\tau(t))$, etc.

1 at most one job, and each vacancy concerns exactly one worker; once a match
 2 has been made, the governmental job bureau removes the matched pair from its
 3 ledger, and the process continues.

4 For concreteness, we shall next define a specific family of match cost distributions.
 5 This is done solely to streamline our presentation: Our methodology allows us
 6 to be more general as we discuss in Section 6. Suppose that the quality of a
 7 (candidate) pair is characterized by an inherent *match parameter* λ_{ij} where a
 8 higher parameter will represent a lower expected match cost. Match costs are
 9 independently and exponentially distributed with rate λ_{ij} .¹³

10 Specifically, we posit that the *match cost* $w_{ij} > 0$ when client $i \in \mathcal{C}$ is matched
 11 to provider $j \in \mathcal{P}$ is an independent draw from an exponential distribution of
 12 rate λ_{ij} for any time τ , that is, $w_{ij} \sim \exp(\lambda_{ij})$. For example, a popular model
 13 assumes that λ_{ij} is composed by additively separable components describing the
 14 agents' types and a couple-specific term depending possibly on both the identity
 15 of the agents and their types (Kanoria et al., 2018). For generality, our only
 16 assumption regarding the rate parameters λ_{ij} is that they are bounded from below
 17 by $\underline{\lambda}$, from above by $\bar{\lambda}$, and have mean value λ , that is, $\lambda = \lim_{\tau \rightarrow \infty} [N_{\mathcal{C}}(\tau) +$
 18 $N_{\mathcal{P}}(\tau)]^{-1} \sum_{i=1}^{N_{\mathcal{C}}(\tau)} \sum_{j=1}^{N_{\mathcal{P}}(\tau)} \lambda_{ij}$.

19 **The social planner.** Throughout the sequel, we assume the existence of a so-
 20 cial planner who, whenever an agent arrives on the market, observes the arrival;
 21 other than that, the social planner has no other information regarding the arrival
 22 process of the agents (or the distribution of their match costs). Due to this lack of
 23 information, the social planner has no basis to judge whether a particular agent
 24 arriving in the market is 'good' or 'bad', and is thus left with the challenge of
 25 choosing a clearing schedule with which to operate the market. In the sequel, we
 26 will also write $A \equiv A(\tau)$ for the number of clients/providers that have been as-
 27 signed a partner up to time τ , and $R(\tau) = N_{\mathcal{C}}(\tau) + N_{\mathcal{P}}(\tau) - 2A(\tau)$ for the number
 28 of unmatched agents up to time τ .

29 With all this in hand, a *clearing schedule* (CS) will be a rule that determines:

- 30 (i) At which points in time $\tau \in (0, \infty)$ to trigger a *matching event* (ME), possibly
 31 depending on $N_{\mathcal{C}}(\tau)$, $N_{\mathcal{P}}(\tau)$ and $A(\tau)$.

¹³As mentioned by Aldous (2001) and developed in detail by Janson (1999, Section 2) generalizations to larger classes of distributions are easily obtained. For ease of exposition we stick to exponential distributions with the exception of Section 6 that generalizes our main results.

1 (ii) Which players to match at a given matching event, possibly depending on
 2 the current match costs of agents who have already arrived to the market
 3 until time τ .

4 After a matching event, the players who are being matched leave the market, while
 5 the unmatched players remain on the market.

6 In what follows, we shall focus on clearing schedules that match a single couple
 7 per matching event. In particular, the clearing schedules we analyze will match
 8 the couple with the minimal matching cost in each matching event.¹⁴ Restricting
 9 ourselves to these kinds of clearing schedules is motivated by our aim to study
 10 clearing schedules that can be paired with a market mechanism (e.g., a two-sided
 11 auction). Myerson and Satterthwaite (1983) and Rustichini et al. (1994) show the
 12 general impossibility to have ex-post efficient and budget balanced mechanisms
 13 for two-sided market games with private information. Their results rely on the
 14 assumption that, with positive probability, any given client-provider pair have
 15 valuations for each other such that trade is not individually rational for both at
 16 any price. Since the latter doesn't hold in our setting, one could micro-found
 17 the interaction avoiding the impossibility, that is, define a mechanism that is ex-
 18 post efficient and budget balanced. We shall assume throughout the analysis the
 19 existence of such a mechanism; however, given our focus on the social planner an
 20 explicit analysis of such mechanisms is beyond the scope of the present paper.

21 Now, as discussed before, the social planner aims to match clients and providers
 22 optimally along two axes: *a*) to reduce the coexistence of clients and providers
 23 (i.e., *waiting time*); and *b*) to match clients and providers in a way that minimizes
 24 matching cost (i.e., *mismatch*). Beginning with the latter, the *expected matching*
 25 *cost* for the first A couples is defined as

$$\text{cost}_{\text{CS}}(A) \equiv \mathbb{E} \left[\sum_{k=1}^A w_{i_k, j_k} \right] \quad (1)$$

26 where w_{i_k, j_k} is the match cost of the k -th matched couple and the expectation is
 27 taken with respect to the random arrival of clients and providers and the random-
 28 ness of the match costs. Similarly, the *expected waiting time* of a clearing schedule

¹⁴The only clearing schedule that we consider and which violates this principle is the first-come, first-served (FCFS) clearing schedule which we describe in Section 3.

1 until time T is defined as

$$\text{wait}_{\text{CS}}(T) \equiv \mathbb{E} \left[\int_0^T R(\tau) d\tau \right] \quad (2)$$

2 where the expectation is taken with respect to the random arrival of agents in
3 the market. In the sections that follow, we will explore the optimization of these
4 two performance metrics, and the trade-offs that arise when trying to minimize
5 both.

6 **3 The trade-off between waiting time and match-** 7 **ing costs**

8 Our analysis begins with the case of a single-minded social planner. Specifically,
9 we investigate which clearing schedule a social planner would employ if either
10 only caring about the expected waiting time, or only caring about the expected
11 matching cost. After we deal with these two cases separately, we shall proceed to
12 show that these objectives are mutually incompatible and lead to an unavoidable
13 trade-off for the social planner.

14 **Single-minded social planners.** First, a social planner who is optimizing the
15 agents' expected waiting time will choose a clearing schedule which leaves no un-
16 matched couples at any point in time. To do so, we will consider a 'greedy' clearing
17 schedule, denoted $\text{CS}_{\text{greedy}}$, which performs a minimum weight matching whenever
18 there is exactly one unmatched client/provider pair in the market. Second, a so-
19 cial planner who is optimizing the agents' expected matching cost will choose a
20 clearing schedule which – ideally – waits until everyone has arrived in the market
21 and then matches agents optimally (thus minimizing the sum of match costs).¹⁵
22 That is, the hypothetical 'patient' clearing schedule, denoted $\text{CS}_{\text{patient}}$, should be
23 preferred by any social planner who is only concerned with the expected matching
24 cost.

25 The implementation of these schedules leads to the following matching cost and
26 waiting time:

¹⁵To make such a clearing schedule realistic, all agents would need to arrive in the market in finite time; since this schedule will mostly serve as a theoretical comparison baseline, we will not consider this issue in detail.

1 **Proposition 1.** *The optimal clearing schedules for a single-objective social plan-*
 2 *ner are:*

3 (1) *The patient clearing schedule $\text{CS}_{\text{patient}}$ is optimal with respect to matching cost*
 4 *minimization; in particular, for all $A \geq 1$, we have:*

$$\frac{\log 2}{\lambda} \leq \text{cost}_{\text{patient}}(A) \leq \frac{\pi^2}{6\lambda}$$

5 (2) *The greedy clearing schedule $\text{CS}_{\text{greedy}}$ is optimal with respect to waiting time*
 6 *minimization; in particular, for all $\tau \geq 0$, we have:*

$$\text{wait}_{\text{greedy}}(\tau) = \frac{2}{3}\tau^{3/2}$$

7 *Remark.* In view of [Proposition 1](#), the expected matching cost of $\text{CS}_{\text{patient}}$ and the
 8 expected waiting time of $\text{CS}_{\text{greedy}}$ will serve as the benchmark for comparing the
 9 matching cost and waiting time of any other clearing schedule.

10 *Proof of Proposition 1.* We prove our claims for each of the two clearing schedules
 11 separately.

12 *Part 1: Matching cost minimization.* For the first assertion, note that the expo-
 13 nential distribution is closed under scaling by a positive factor, i.e., if $X \sim \exp(\kappa)$
 14 then $\mu X \sim \exp(\kappa/\mu)$. In our case, this implies that

$$w_{ij} \sim \exp(\lambda_{ij}) \iff w_{ij} \sim \frac{1}{\lambda_{ij}} \exp(1) \quad (3)$$

15 We have that for all $i \in \mathcal{C}$, $j \in \mathcal{P}$, the distribution of w_{ij} is first-order stochastically
 16 dominated by $\underline{\lambda}^{-1} \exp(1)$. Thus the expected weight is upper bounded by the
 17 simplified problem where all match costs are distributed according to $\underline{\lambda}^{-1} \exp(1)$.
 18 With this in mind, we will simplify notation in the rest of the proof by setting
 19 $\underline{\lambda} = 1$.

20 By the summation formula of [Buck et al. \(2002\)](#) and [Linusson and Waestlund](#)
 21 [\(2004\)](#), we have for the expected weight of the minimum A -matching (note that
 22 $A = N$, recalling that $N = \min\{N_{\mathcal{C}}, N_{\mathcal{P}}\}$):

$$\mathbb{E}_{\min} \left[\sum_{k=1}^N w_{i_k, j_k} \right] = \sum_{\substack{i, j \geq 0 \\ i+j < N}} \frac{1}{(N_{\mathcal{C}} - i) \cdot (N_{\mathcal{P}} - j)}. \quad (4)$$

1 Thus, we readily have

$$\text{cost}_{\text{patient}}(N) = \mathbb{E}_{\min} \left[\sum_{k=1}^N w_{i_k, j_k} \right] \leq \sum_{\substack{i, j \geq 0 \\ i+j < N}} \frac{1}{(N-i) \cdot (N-j)}. \quad (5)$$

2 To proceed, by Waestlund (2009, Lemma 3.1) we have

$$\sum_{\substack{i, j \geq 0 \\ i+j < N}} \frac{1}{(N-i) \cdot (N-j)} = \sum_{k=1}^N \frac{1}{k^2} \leq \zeta(2), \quad (6)$$

3 where $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ is the Basel constant. Returning to our original
4 problem, we conclude that $\text{cost}_{\text{patient}}(A) \leq \pi^2/(6\lambda)$, as claimed.

5 To compute the lower bound, we need to consider the expected match cost for
6 $A = 1$, because matching more players would only serve to increase the expected
7 matching cost. In the patient clearing schedule for $A = 1$, the process terminates
8 when at least one client and at least one provider have entered the market. Let
9 $Y \geq 2$ be the number of agents required to observe at least one client and one
10 provider. Then the event $Y = k + 1$ is the same event as the union of the disjoint
11 events ‘the first k agents are clients and the $(k + 1)$ -th agent is a provider’ and ‘the
12 first k agents are providers and the $(k + 1)$ -th agent is a client’. Each of the latter
13 events has probability $1/2^{k+1}$, so $\mathbb{P}(Y = k + 1) = 2^{-k}$. Moreover, as we prove in
14 Lemma 8, for $Y = k + 1$, the expected minimum match cost is given by $\frac{1}{\lambda \cdot k}$. Thus
15 for $A = 1$, we get

$$\text{cost}_{\text{patient}}(N) = \sum_{k=1}^{\infty} \frac{1}{\lambda k} \mathbb{P}(Y = k + 1) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{1}{2^k k} = \frac{\log 2}{\lambda}, \quad (7)$$

16 where the last equality follows from the series expansion $\log(1 - x) = -x - x^2/2 -$
17 $x^3/3 - \dots$ applied to $x = 1/2$.

18 *Part 2: Waiting time minimization.* For our second assertion, note that, at any
19 point in time, there are either no clients or no providers in the market. In view
20 of this, let S_τ be the difference of clients and providers who have arrived to the
21 market until time τ , that is, $S_\tau = N_C(\tau) - N_P(\tau)$. Then, for all $T > 0$, we get:

$$\text{wait}_{\text{greedy}}(T) = \mathbb{E} \left[\int_0^T |S_\tau| d\tau \right] = \int_0^T \mathbb{E}[|S_\tau|] d\tau \quad (8)$$

1 where the latter equality holds by Tonelli's theorem (since $|S_\tau|$ is non-negative).
 2 Applying Tonelli's theorem a second time, we can consider the case where the
 3 expectation with respect to the arrival times is taken first. To do so, consider
 4 the process where at the fixed points in time $\tau = 1, 2, \dots$ an agent arrives to the
 5 market and let \bar{S}_τ be the difference of clients and providers who have arrived to
 6 the market at time τ . We then have:

$$\mathbb{E} \left[\int_0^T |S_\tau| d\tau \right] = \int_0^T \mathbb{E}[|S_\tau|] d\tau = \int_0^T \mathbb{E}[|\bar{S}_\tau|] d\tau \quad (9)$$

It is well-known that for $\tau \rightarrow \infty$ the appropriately rescaled random walk \bar{S}_τ converges in distribution to the Wiener process W_τ (Kac, 1947). Thus, for large T , Eq. (9) gives

$$\mathbb{E} \left[\int_0^T |S_\tau| d\tau \right] = \int_0^T \mathbb{E}[|W_\tau|] d\tau = \int_0^T \sqrt{\text{Var}(W_\tau)} d\tau = \int_0^T \sqrt{\tau} d\tau = \frac{2}{3} T^{3/2}. \quad \square$$

7

8 This concludes the analysis of a single-minded social planner who either only cares
 9 about the expected waiting time, or only the expected matching cost.

10 **Multi-objective social planners.** Going beyond the narrow view of a single-
 11 minded social planner, we proceed below to examine the case of social planners
 12 that care about *both* the expected cost of matching and the agents' overall expected
 13 waiting time. Natural candidates to evaluate a clearing schedule in this context
 14 are the expected matching ratio and the expected waiting ratio, defined below as
 15 follows:

1. The *expected matching ratio* of a clearing schedule CS is

$$\alpha \equiv \alpha(A) = \frac{\text{cost}_{\text{CS}}(A)}{\text{cost}_{\text{patient}}(A)}. \quad (10)$$

2. The *expected waiting ratio* of a clearing schedule CS is

$$\beta \equiv \beta(\tau) = \frac{\text{wait}_{\text{CS}}(\tau)}{\text{wait}_{\text{greedy}}(\tau)}. \quad (11)$$

16 Note that $\text{CS}_{\text{patient}}$ is optimal with respect to expected matching cost while $\text{CS}_{\text{greedy}}$
 17 is optimal with respect to expected waiting time.

1 Going forward, note that all candidate clearing schedules can be characterized
2 by a function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that the k -th ($k \in \mathbb{N}$) couple is matched when
3 $\lceil f(k) \rceil$ agents are on the short side of the market. Denote a clearing schedule that
4 is defined via a function f by CS_f . Without any restrictions on f this includes all
5 possible clearing schedules that always match the couple with the minimal match
6 cost. However, given that we wish to analyze the asymptotic regime where enough
7 agents have entered the market, we focus below on a natural class of functions
8 introduced by Hardy (1910) which make such comparisons possible. Specifically,
9 each function in this class is defined, for all $x \geq 0$, by a finite combination of
10 the basic arithmetic operations (addition, multiplication, raising to a power, and
11 their inverses), operating on the variable x and on real constants. Hardy (1910,
12 Theorem, page 18) shows that for any two such functions, f and g , either $f =$
13 $\omega(g)$, $f = \Theta(g)$, or $f = o(g)$.

14 Then, for such functions, we will use the following asymptotic notations:
15 $f(x) = \mathcal{O}(g(x))$ if $f(x) < c \cdot g(x)$ for some $c > 0$ constant and x sufficiently large.
16 $f(x) = \Omega(g(x))$ is the inverse \mathcal{O} notation ($f(x) > c \cdot g(x)$ for x sufficiently large).
17 $f(x) = \Theta(g(x))$ if there exist two constants $k, K \geq 0$ and a positive integer x_0
18 such that $kg(x) \leq f(x) \leq Kg(x)$ for all $x \geq x_0$.
19 For $g(x)$ non-zero $f(x) = o(g(x))$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$ and $f(x) = \omega(g(x))$ if
20 $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$.

21 In light of the above, the first question that arises is whether there exists a clear-
22 ing schedule that is optimal along *both* axes (at least, asymptotically). To formalize
23 this, we say that a clearing schedule CS has *finite expected matching ratio*
24 if $\limsup_{A(\tau) > 0} \alpha(A(\tau)) < \infty$; likewise, we say that clearing schedule has *finite*
25 *expected waiting ratio* if $\limsup_{\tau > 0} \beta(\tau) < \infty$.

26 The following theorem shows that the answer to the above question is a resounding
27 ‘no’:

28 **Theorem 2** (‘No free lunch’). *There exists no clearing schedule simultaneously*
29 *achieving finite ratios for both expected matching cost and waiting time.*

30 **Theorem 2** illustrates that multi-objective social planners are faced with a crucial
31 trade-off independently of their specific utility function – provided of course that
32 they care about both matching cost and waiting time in a non-trivial way. In
33 addition, **Theorem 2** justifies the performance measures for the two dimensions
34 of mismatch (expected matching ratio and waiting ratio) and in particular the
35 sufficiency to analyze them in terms of orders of τ (or $A = A(\tau)$).

1 *Proof of Theorem 2.* We prove this result by contradiction; specifically, we find a
 2 necessary condition for a clearing schedule to have finite expected matching ratio
 3 and then show that clearing schedules satisfying this condition cannot have a finite
 4 expected waiting ratio.

5 To make this precise, consider the clearing schedule CS_f that matches the k -th
 6 couple when $N - (k - 1) = \lceil f(k) \rceil$. We shall show that a necessary condition for
 7 a clearing schedule CS_f to have finite expected matching ratio is

$$f(k) = \omega(k^{1/2}) \quad (12)$$

8 To show this, consider the clearing schedule that matches the k -th couple when
 9 at least $\lceil k^{1/2} \rceil$ players are on the short side of the market; with a fair degree of
 10 hindsight, denote this clearing schedule as $\text{CS}_{\gamma=1/2}$.¹⁶ As we show in [Theorem 3](#),
 11 this clearing schedule has $\alpha(A) = \Theta(\log A)$. Thus, in order for another schedule
 12 CS_f to have finite expected matching ratio, matching events have to happen orders
 13 of magnitude later than in $\text{CS}_{\gamma=1/2}$. Concretely, for k large enough the k -th couple
 14 is cleared at time $\tau_{\text{CS}_f}(k) = \omega(\tau_{\text{CS}_{\gamma=1/2}}(k))$. It follows that [Eq. \(12\)](#) holds.

15 We can now analyze the expected waiting time for CS_f such that [Eq. \(12\)](#) holds
 16 for f . To construct a lower bound, consider the alternative arrival process, where
 17 clients and providers alternately arrive to the market. Note that for any given
 18 clearing schedule this process incurs lower waiting time. For the clearing schedule
 19 we consider the waiting time of this alternative arrival process is precisely governed
 20 by the fact that the k -th match takes place when at least $f(k)$ players are on the
 21 short side of the market. Further, note that $\tau(A)$ is clearly smaller for this new
 22 arrival process compared to the original process. Given that we only need to show
 23 that the waiting time is increasing in A it suffices to show that it is increasing in
 24 τ (not conditioned on A). Thus, the waiting time is lower bounded by using the
 25 approximation by the Wiener process (as in the proof of [Proposition 1](#)) and by
 26 observing that arrival is governed by a Poisson clock of rate 1:

$$\int_0^T 2f(\tau)d\tau = \omega\left(\int_0^T 2\sqrt{\tau} d\tau\right) = \omega(T^{3/2}) \quad (13)$$

27 By [Proposition 1\(ii\)](#), the optimal expected waiting time is $(2/3)T^{3/2}$, so we con-
 28 clude that the expected waiting ratio is lower bounded by $\omega(T^{3/2})/T^{3/2} = \omega(1)$
 29 and our proof is complete. \square

¹⁶See [Section 4](#) for detailed definitions.

1 This concludes our first result for multi-objective social planners, showing that
 2 the trade-off between cost of matching and waiting time is essential.

3 4 Interpolating between waiting time and match- 4 ing cost

5 In this section, we analyze a class of clearing schedules covering a broad spectrum
 6 of social planning desiderata interpolating between matching cost and waiting
 7 time.

8 To begin, recall that [Proposition 1](#) provides the expected matching cost of the
 9 patient clearing schedule $\text{CS}_{\text{patient}}$ (which minimizes mismatches) and the expected
 10 waiting time of the greedy clearing schedule $\text{CS}_{\text{greedy}}$ (which minimizes waiting
 11 times). Interpolating between these two ‘extreme’ schedules, we shall consider
 12 below a class of clearing schedules where the social planner waits for some length
 13 of time in order to accrue some intermediate number of agents on both sides of
 14 the market. Concretely, we shall study clearing schedules that match the k -th
 15 couple when $N - k = f(k)$, i.e., when $f(k)$ players are on the short side of the
 16 market.¹⁷

For concreteness, we restrict ourselves to clearing schedules of the form

$$f(k) = \Theta(k^\gamma) \quad \text{for some } \gamma \in [0, 1]. \quad (14)$$

17 For $\gamma = 0$, the induced clearing schedules match players once a constant threshold
 18 is reached; in particular, the greedy schedule is recovered when $f \equiv 1$ (correspond-
 19 ing to $\gamma = 0$). More generally, we shall denote clearing schedules of the above form
 20 by CS_γ and write $\text{CS}_{\gamma=1/2}$ for the clearing schedule with $\gamma = 1/2$. Similarly we shall
 21 use the notation α_γ for the expected matching ratio of CS_γ and β_γ for the expected
 22 waiting ratio of CS_γ .

23 In addition to the clearing schedules induced by the assumptions above, we shall
 24 also consider another natural schedule based on the principle of *first-come, first-*
 25 *served* (FCFS), i.e., when agents are matched as soon as possible on a first-come,
 26 first-served basis. This schedule, which we denote by CS_{FCFS} , differs from $\text{CS}_{\text{greedy}}$
 27 in terms of who is matched with whom (first-come, first-served vs. minimum cost
 28 matching) but not regarding when a matching event occurs. As such, given that

¹⁷Recall that $N = \min\{N_C, N_P\}$.

1 CS_{FCFS} does not take into account matching costs, it is not reasonable to expect
 2 that it will perform well on any dimension other than the agents' expected waiting
 3 times. On the other hand, it exhibits 'fairness' relative to the agents' arrival times,
 4 a feature which is crucial in many applications.¹⁸

5 **Overview of results.** Table 1 summarizes all clearing schedules analyzed below
 6 (including a 'balanced' schedule, $\text{CS}_{\text{balanced}}$, that we discuss in Section 5). Our
 7 results (in terms of each schedule's expected matching and waiting ratio) are
 8 then summarized in Table 2: as can be seen, the family of schedules under study
 9 captures the full range between schedules that are 'good' relative to mismatches
 10 and 'bad' relative to waiting times, and vice versa.

CS_{FCFS}	match players as soon as possible on a first-come, first-served basis
$\text{CS}_{\text{greedy}}$	match players as soon as possible
CS_{γ}	match the k -th couple when $\Theta(k^{\gamma})$ players are on the short side of the market ($0 \leq \gamma \leq 1$)
$\text{CS}_{\text{patient}}$	match players optimally after everyone has arrived
$\text{CS}_{\text{balanced}}$	match the k -th couple when $\Theta(k^{1/2}(\log k)^{1/3})$ players are on the short side of the market.

Table 1: Overview of the various clearing schedules considered in the sequel. Note that all schedules other than CS_{FCFS} match the couple with the minimum match cost at each matching event.

11 In view of these results, the clearing schedule $\text{CS}_{\gamma=1/2}$ can be seen as a *phase tran-*
 12 *sition* between two markedly different regimes. On the one hand, for $\gamma < 1/2$,
 13 the expected matching ratio $\alpha(A)$ grows as a power law in A while the expected
 14 waiting ratio $\beta(\tau)$ is finite. On the other hand, for $\gamma > 1/2$, we have a finite
 15 expected matching ratio but an expected waiting ratio that grows polynomially.
 16 Finally, at the critical point $\gamma = 1/2$, the expected matching ratio grows to in-
 17 finity for large A , but at a slow, logarithmic rate ($\Theta(\log A)$). Notably, the phase
 18 transition at $\gamma = 1/2$ signifies a discontinuity of the expected matching ratio, so it
 19 is a *first-order* phase transition; by contrast, the expected waiting ratio exhibits
 20 no such discontinuity, signifying a *second-order* phase transition.

21 The infinite matching ratio vis-a-vis the finite waiting ratio for $\gamma = 1/2$ suggests

¹⁸Indeed, this may be a desirable feature in applications such as processor time requests in distributed computing. We shall leave extensions of our analyses to include fairness considerations for future work.

SCHEDULE	DESCRIPTION	MATCHING RATIO, α	WAITING RATIO, β
CS_{FCFS}	FCFS matching	$\Theta(A)$	1
$\text{CS}_{\text{greedy}}$	Greedy matching	$\Omega(A^{1/2})$	1
$\text{CS}_{0 \leq \gamma < 1/2}$	Subcritical rate matching	$\Omega(A^{1/2-\gamma})$	$\Theta(1)$
$\text{CS}_{\gamma=1/2}$	Critical rate matching	$\Theta(\log A)$	$\Theta(1)$
$\text{CS}_{1/2 < \gamma \leq 1}$	Supercritical rate matching	$\Theta(1)$	$\Theta(\tau^{\gamma-1/2})$
$\text{CS}_{\text{patient}}$	Patient matching	1	$\Theta(\tau^{1/2})$
$\text{CS}_{\text{balanced}}$	Balanced matching	$\Theta((\log A)^{1/3})$	$\Theta((\log A)^{1/3})$

Table 2: The range of expected matching and waiting ratios; $\text{CS}_{\text{balanced}}$ is discussed in Section 5. Recall from Eq. (10) and Eq. (11) the expected matching ratio $\alpha \equiv \alpha(A) = \frac{\text{cost}_{\text{CS}}(A)}{\text{cost}_{\text{patient}}(A)}$ ($\text{cost}_{\text{patient}}(A) = \Theta(1)$) and the expected waiting ratio $\beta \equiv \beta(\tau) = \frac{\text{wait}_{\text{CS}}(\tau)}{\text{wait}_{\text{greedy}}(\tau)}$ ($\text{wait}_{\text{greedy}}(\tau) = \Theta(\tau^{3/2})$).

1 that further fine-tuning should be possible and, indeed, the ‘balanced’ schedule
2 $\text{CS}_{\text{balanced}}$ (which we define and discuss in Section 5) reduces the growth of the
3 expected matching ratio by a factor of $(\log A)^{2/3}$ while increasing the expected
4 waiting ratio $\beta(\tau)$ by a factor of $(\log \tau)^{1/3}$. In a sense (that we shall make precise in
5 the next section) this is as close as we can get to a ‘free lunch’ in this setting.

6 **Formal statements.** We now proceed to provide complete statements of the
7 results discussed above. To streamline our presentation, we have relegated the
8 detailed proofs to Appendices A and B; however, the main pattern of the proofs
9 can also be seen in Section 5 where we treat the case of $\text{CS}_{\text{balanced}}$.

10 We begin with our results for the matching cost ratio α :

Theorem 3. *The expected matching ratios for the schedules under study are as follows:*

$$(CS_{\text{FCFS}}) \quad \text{FCFS matching:} \quad \alpha_{\text{FCFS}} = \frac{6\lambda}{\pi^2} A \quad (15a)$$

$$(CS_{\text{greedy}}) \quad \text{Greedy matching:} \quad \alpha_{\text{greedy}} \geq \frac{6\lambda}{5\pi^2} A^{1/2} \quad (15b)$$

$$(CS_{0 \leq \gamma < 1/2}) \quad \text{Subcritical rate matching:} \quad \underline{C}_\gamma A^{1/2-\gamma} \leq \alpha_{0 \leq \gamma < 1/2} \leq \bar{C}_\gamma A^{1-2\gamma} \quad (15c)$$

$$(CS_{\gamma=1/2}) \quad \text{Critical rate matching:} \quad \frac{2\lambda}{\pi^2} \log A \leq \alpha_{\gamma=1/2} \leq \frac{\bar{\lambda} (1 + \log A)}{\lambda \log 2} \quad (15d)$$

$$(CS_{1/2 < \gamma \leq 1}) \quad \text{Supercritical rate matching:} \quad \alpha_{1/2 < \gamma \leq 1} = \frac{\bar{\lambda} \zeta(2\gamma)}{\lambda \log 2} \quad (15e)$$

$$(CS_{\text{patient}}) \quad \text{Patient matching:} \quad \alpha_{\text{patient}} = 1 \quad (15f)$$

- 1 *Remark.* In the above, \underline{C}_γ and \bar{C}_γ are positive constants, and $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$
- 2 denotes the Riemann zeta function (so $\zeta(s) < \infty$ for all $s > 1$).
- 3 By contrast, for the expected waiting ratio β , we have:

Theorem 4. *The expected waiting ratios for the schedules under study are as follows:*

$$(CS_{\text{FCFS}}) \quad \text{FCFS matching:} \quad \beta_{\text{FCFS}} = 1 \quad (16a)$$

$$(CS_{\text{greedy}}) \quad \text{Greedy matching:} \quad \beta_{\text{greedy}} = 1 \quad (16b)$$

$$(CS_{0 \leq \gamma < 1/2}) \quad \text{Subcritical rate matching:} \quad \beta_{0 \leq \gamma < 1/2} = \Theta(1) \quad (16c)$$

$$(CS_{\gamma=1/2}) \quad \text{Critical rate matching:} \quad \beta_{\gamma=1/2} = \Theta(1) \quad (16d)$$

$$(CS_{1/2 < \gamma \leq 1}) \quad \text{Supercritical rate matching:} \quad \beta_{1/2 < \gamma \leq 1} = \Theta(\tau^{\gamma-1/2}) \quad (16e)$$

$$(CS_{\text{patient}}) \quad \text{Patient matching:} \quad \beta_{\text{patient}} = \Theta(\tau^{1/2}) \quad (16f)$$

- 4 In closing this section, it is worth noting that the bounds for α become asymptot-
- 5 ically ‘less tight’ for small $\gamma < \frac{1}{2}$. As far as this gap is concerned, we conjecture
- 6 that the upper bound is the tight one: the lower bound is obtained via a crude
- 7 approximation using Jensen’s inequality, and this could be potentially tightened
- 8 (although we haven’t been able to do so). By contrast, the approximation for the

1 upper bound seems less drastic.

2 We should also note that the results in [Theorem 4](#) are driven by the assumption
3 that the arrival of either a client or a provider at every stage of the process is
4 equally likely. This entails that the expected absolute difference of clients and
5 providers $|N_{\mathcal{C}}(\tau) - N_{\mathcal{P}}(\tau)|$ can be approximated by a Wiener process as detailed
6 in [Appendix B](#). For the latter we know that the expectation is $\sqrt{\tau}$, so $|N_{\mathcal{C}}(\tau) -$
7 $N_{\mathcal{P}}(\tau)| \approx \sqrt{\tau}$ in expectation. It would be interesting to consider different arrival
8 processes such as an urn model with delayed replacement where $|N_{\mathcal{C}}(\tau) - N_{\mathcal{P}}(\tau)|$
9 would exhibit a different asymptotic behavior; we leave this analysis to future
10 work.

11 5 A balanced social planner

12 Until now, we have analyzed clearing schedules based on the trade-off between
13 waiting time and matching cost, but without explicitly comparing the two. In this
14 section, we shall commit to a specific class of utility functions in order to make an
15 explicit comparison between these otherwise incomparable quantities.

16 To that end, let $u(\cdot)$ denote the expected utility (or ‘welfare’) of the social planner
17 given a specific clearing schedule. Assume further that the functions expressing
18 this utility depend on both the expected matching cost and the expected waiting
19 time via the additively separable expression

$$u(\text{CS}) = u_{\text{cost}}(\alpha_{\text{CS}}) + u_{\text{wait}}(\beta_{\text{CS}}) \quad (17)$$

20 In order to make comparisons between the utility components u_{cost} and u_{wait} we
21 shall first consider their respective maximum values. It is then natural to assume
22 that u_{cost} is maximal for the patient clearing schedule (which minimizes matching
23 cost) and that u_{wait} is maximal for the greedy clearing schedule (which minimizes
24 waiting time). We shall thus assume that the two maxima are of the same order,
25 viz.,

$$\sigma \cdot u_{\text{cost}}(\alpha_{\text{patient}}) = (1 - \sigma) \cdot u_{\text{wait}}(\beta_{\text{greedy}}) \quad (18)$$

26 where $\sigma \in (0, 1)$ is a constant factor that specifies the relative importances of the
27 disutilities from mismatching versus waiting. Naturally, we require that the social
28 planner seeks to minimize both the costs of matching and the agents’ waiting time.
29 As such, we make the assumption that u_{cost} is a concave function that decreases
30 in the expected matching cost, and u_{wait} is a concave function that decreases in

1 the expected waiting time.

2 In view of all this, a social planner is said to be *balanced* if the disutilities from
 3 mismatching and waiting display similar growths for large τ . That is, for a given
 4 clearing schedule \mathbf{CS} with expected matching ratio α and expected waiting time
 5 β , we assume that

$$u_{\text{cost}}(\alpha_{\mathbf{CS}}) = \Theta(u_{\text{wait}}(\beta_{\mathbf{CS}})) \quad \text{whenever} \quad \alpha_{\mathbf{CS}} = \Theta(\beta_{\mathbf{CS}}) \quad (19)$$

6 In this general context, we obtain the following result governing balanced social
 7 planning:

8 **Theorem 5.** *Let $\mathbf{CS}_{\text{balanced}}$ be the clearing schedule that matches the k -th couple
 9 when $N - (k - 1) = \lceil k^{1/2}(\log k)^{1/3} \rceil$ players are on the short side of the mar-
 10 ket. The expected matching and waiting ratios incurred by $\mathbf{CS}_{\text{balanced}}$ are both
 11 $\Theta((\log A)^{1/3})$; moreover, any other schedule \mathbf{CS}_f achieving this balance has $f(k) =$
 12 $\Theta(k^{1/2}(\log k)^{1/3})$.*

13 *Remark.* For technical reasons we state our result in terms of the number of
 14 matched couples ($A = A(\tau)$). Note that, for any clearing schedule where the pro-
 15 portion of matched players increases over time (more precisely, where $\lim_{\tau \rightarrow \infty} \frac{A(\tau)}{N_{\mathcal{C}}(\tau) + N_{\mathcal{P}}(\tau)} =$
 16 1), A is growing at the same rate as τ .

17 *Proof of Theorem 5.* Consider a clearing schedule of the form \mathbf{CS}_f that matches
 18 the k -th couple when $\lceil f(k) \rceil$ players on the short side of the market. In order
 19 to balance the expected matching and waiting ratios, any such clearing schedule
 20 would have to satisfy $f(k) = \omega(\sqrt{k})$; otherwise, the expected matching ratio would
 21 dominate asymptotically the expected waiting ratio (see Table 2). Thus, without
 22 loss of generality, we can assume that $f(k)$ is non-decreasing for large k .

23 Let $t(k, f(k))$ be the stopping time for the event that for the k -th time at least
 24 $f(k)$ clients and $f(k)$ providers are in the market, assuming that every time this
 25 is the case one client and one provider are removed. Finally, recall that $S_\tau =$
 26 $N_{\mathcal{C}}(\tau) - N_{\mathcal{P}}(\tau)$ is the difference of clients and providers who have arrived to the
 27 market until τ .

We begin with the expected matching ratio. For the upper bound we have:

$$\begin{aligned} \mathbb{E} \left[\sum_{k=1}^A \frac{1}{(\lceil f(k) \rceil + |S_{t(k, f(k))}|) \lceil f(k) \rceil} \right] &= \sum_{k=1}^A \mathbb{E} \left[\frac{1}{(\lceil f(k) \rceil + |S_{t(k, f(k))}|) \lceil f(k) \rceil} \right] \\ &\leq \sum_{k=1}^A \lceil f(k) \rceil^{-2} \end{aligned} \quad (20)$$

For the lower bound, an algebraic argument which we make precise in [Appendix A](#) (cf. [Eq. \(58\)](#)) shows that $\mathbb{E}[t(k, f(k))] < 10k$. Furthermore, note that $\mathbb{E}[S_t]$ is strictly increasing in t . Thus, by Jensen's inequality, and [Lemma 9](#) (which is bounding $|S_t|$ via a combinatorial argument and using Stirling's formula), we get:

$$\begin{aligned} \sum_{k=1}^A \mathbb{E} \left[\frac{1}{(\lceil f(k) \rceil + |S_{t(k, f(k))}|) \lceil f(k) \rceil} \right] &\geq \sum_{k=1}^A \frac{1}{(\lceil f(k) \rceil + \mathbb{E}[|S_{10k}|]) \lceil f(k) \rceil} \\ &\geq \frac{\pi}{\sqrt{2}e} \sum_{k=1}^A \frac{1}{(\lceil f(k) \rceil + \sqrt{10k}) \lceil f(k) \rceil} \\ &= \Theta \left(\sum_{k=1}^A \frac{1}{f(k)^2} \right) \end{aligned} \quad (21)$$

1 where the last line follows from the assumption $f(k) = \omega(k^{1/2})$. Thus the two
2 bounds together with the fact that the patient schedule has finite matching cost
3 yield the result that, for $f(k) = \omega(k^{1/2})$ the expected matching ratio is $\alpha(A) =$
4 $\Theta \left(\sum_{k=1}^A 1/f(k)^2 \right)$.

5 We proceed, by considering the incurred waiting time. Recall that $N(\tau) - A(\tau)$
6 is the number of agents on the shorter side of the market at time τ , so $N - A =$
7 $\lceil f(k-1) \rceil - 1$ after the $(k-1)$ -st match. The number of clients and the number
8 of providers that need to arrive to the market before the k -th match is thus upper
9 bounded by

$$\lceil f(k) \rceil - (\lceil f(k-1) \rceil - 1) = 1 + \lceil f(k) \rceil - \lceil f(k-1) \rceil \leq 2 \quad (22)$$

10 where the last inequality follows from the fact that $f(k) = o(k)$ is a necessary
11 condition for a feasible clearing schedule (that is, a clearing schedule where the
12 proportion of unmatched versus matched players is decreasing). The expected
13 waiting time accrued between the $(k-1)$ -st and the k -th match is therefore upper

1 bounded by:

$$\Delta_k := \underbrace{\mathbb{E}[\text{time s.t. } \geq 2 \text{ clients \& } \geq 2 \text{ providers enter market}]}_{=:\Delta_k^1} \cdot \underbrace{(2\lceil f(k)\rceil + \lceil g(k)\rceil)}_{=:\Delta_k^2} \quad (23)$$

2 where $\lceil g(k)\rceil$ is a function that we will use to upper bound $|S_k|$, viz., the random
 3 variable constituting the absolute difference of clients and providers in the market
 4 at time $\tau(k)$. For posterity, note also that Δ_k^1 is the expectation of the time
 5 between the $(k-1)$ -th and the k -th match and Δ_k^2 provides an upper bound for
 6 the number of agents waiting in the time interval between the $(k-1)$ -th and the
 7 k -th match.

8 Given the arrival of agents is governed by a Poisson clock of rate one, we have
 9 $\Delta_k^1 = 5$, i.e., on average, five agents need to enter the market to have at least
 10 two clients and at least two providers. To see this, let Y be the number of flips
 11 of a coin required to observe at least 2 heads (clients) and 2 tails (providers).
 12 The event ' $Y > k$ ' is then equivalent to the union of the events ' $\binom{k}{k-1}$ heads' and
 13 ' $\binom{k}{k-1}$ tails'. The two latter events are disjoint and each has probability $\frac{k}{2^k}$. Thus
 14 $\mathbb{P}[Y > k] = \frac{k}{2^{k-1}}$ and we have

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} \mathbb{P}(Y > k) = 1 + 2 \sum_{k=1}^{\infty} \frac{k}{2^k} = 1 + 2 \sum_{k=1}^{\infty} \sum_{j=1}^k \frac{1}{2^k} \quad (24)$$

$$= 1 + 2 \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \frac{1}{2^k} = 1 + 2 \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} = 5 \quad (25)$$

15 We thus have for Eq. (23)

$$\Delta_k = 5 \cdot (2\lceil f(k)\rceil + \lceil g(k)\rceil) \quad (26)$$

Next, to choose the function $g(k)$, note that the law of the iterated logarithm gives

$$\lim_{k \rightarrow \infty} \frac{|S_k|}{\sqrt{2k \log \log k}} = 1. \quad (27)$$

16 Hence, by choosing $g(k) = \sqrt{2k \log \log k}$, the random variable $|S_k|$ is asymptoti-
 17 cally bounded from above by $g(k)$ with probability one.

18 We consider two cases below, which are exhaustive by Hardy (1910, Theorem,
 19 page 18):

1 **Case 1:** $f(k) = \Omega(g(k))$. For the first case we have:

$$5 \cdot (2f(k) + g(k)) = \Theta(f(k)) \quad (28)$$

2 The expected waiting ratio until A pairs have been matched is bounded from above
 3 by $A^{-3/2} \sum_{k=1}^A \Theta(f(k))$, where we are using the fact that the expected waiting time
 4 for the greedy schedule is given by $A^{3/2}$ (see [Proposition 1](#)). A trivial lower bound
 5 for the expected waiting ratio is then given by

$$\frac{1}{A^{3/2}} \sum_{k=1}^A 2f(k) = \frac{1}{A^{3/2}} \sum_{k=1}^A \Theta(f(k)) \quad (29)$$

6 Thus, the expected waiting ratio is given by

$$\beta(A) = \Theta\left(\frac{1}{A^{3/2}} \sum_{k=1}^A f(k)\right) \quad (30)$$

Moving to the comparison of matching and waiting ratios, we recall that u_{cost} and u_{wait} are decreasing and concave and are of the same order (by assumption). Thus $u = u_{\text{cost}} + u_{\text{wait}}$ is maximized if and only if $\alpha = \Theta(\beta)$. In turn, this holds if and only if

$$\sum_{k=1}^A \frac{1}{f(k)^2} = \Theta\left(\frac{1}{A^{3/2}} \sum_{k=1}^A f(k)\right) \quad (31)$$

or, equivalently, if and only if

$$\int_1^A \frac{1}{f(x)^2} dx = \Theta\left(\frac{1}{A^{3/2}} \int_1^A f(x) dx\right) \quad (32)$$

7 where the asymptotic passage from summation to integration – i.e., from [Eq. \(31\)](#)
 8 to [Eq. \(32\)](#) – is made precise in [Appendix C](#).

9 We shall show that $f(x) = \Theta(\sqrt{x}(\log x)^{1/3})$ is the unique solution to [Eq. \(32\)](#) up to
 10 order. To simplify notation, let $f(x) = \sqrt{x}(\log x)^{1/3}$, so the left-hand side (LHS)
 11 of [Eq. \(32\)](#) becomes

$$\int_1^A \frac{1}{x(\log x)^{2/3}} = 3(\log A)^{1/3} + c \quad (33)$$

where c is uniformly bounded and independent of A . Next, focusing on the RHS

of Eq. (32), we get

$$\begin{aligned}
\frac{1}{A^{3/2}} \int_1^A \sqrt{x}(\log x)^{1/3} dx &= (\log A)^{1/3} - \frac{1}{A^{3/2}} \int_1^A x^{3/2} \frac{1}{3x(\log x)^{2/3}} dx \\
&= (\log A)^{1/3} - \frac{1}{A^{3/2}} \underbrace{\int_1^A \sqrt{x} \frac{1}{3(\log x)^{2/3}} dx}_{=o\left(\int_1^A \sqrt{x} \cdot (\log x)^{1/3} dx\right)} \\
&= \Theta((\log A)^{1/3})
\end{aligned} \tag{34}$$

1 Our uniqueness claim follows by noting that the LHS of Eq. (32) is decreasing
2 in $f(x)$ (in orders of magnitude of the upper bound of the integral) while the
3 right-hand side (RHS) is increasing in $f(x)$.

4 **Case 2:** $f(k) = o(g(k))$. For the second case, assume that $f(k) = o(g(k))$. This
5 implies for the matching cost that¹⁹

$$\int_1^A \frac{1}{f(\tau)^2} d\tau = \omega\left(\int_1^A \frac{1}{g(\tau)^2} d\tau\right) \tag{35}$$

6 The integral on the RHS of Eq. (35) can then be bounded from below as follows

$$\int_1^A \frac{1}{g(\tau)^2} d\tau = \int_1^A \frac{1}{4\tau \log \log \tau} d\tau = \omega\left(\int_1^A \frac{1}{4\tau(\log \tau)^{2/3}} d\tau\right) \tag{36}$$

7 For the integral on the RHS of Eq. (36) we have

$$\int_1^A \frac{1}{4\tau(\log \tau)^{2/3}} d\tau = \Theta((\log A)^{1/3}), \tag{37}$$

Hence, combining these last approximations, we finally get

$$\int_1^A \frac{1}{f(\tau)^2} d\tau = \omega((\log A)^{1/3}). \tag{38}$$

8 Thus any solution satisfying $f(k) = o(g(k))$ (Case 2) has expected matching cost
9 that is $\omega(1)$ relative to the optimal solution. This completes the proof that $f(k) =$
10 $\Theta(\sqrt{x}(\log x)^{1/3})$ is the unique optimal clearing schedules for the balanced social
11 planner. \square

¹⁹Formally, for $\tau \geq e$, the integrand is not well-defined, but the Cauchy principal value of the integral remains finite, and this is the value we are using for $\tau \leq e$. This issue could be side-stepped by shifting the lower limit of the integral to a higher value, but we do not do so in order to simplify the presentation.

1 Note that, up to logarithmic factors, the balanced clearing schedule is close to
 2 the clearing schedule $\text{CS}_{\gamma=1/2}$ which signifies a first-order phase transition for the
 3 expected matching ratio. As discussed earlier, $\text{CS}_{\gamma=1/2}$ only signifies a second-order
 4 phase transition for the expected waiting ratio, thus explaining the gap between
 5 $\text{CS}_{\text{balanced}}$ and $\text{CS}_{\gamma=1/2}$. In practice however, $\text{CS}_{\gamma=1/2}$ seems to be a reasonable
 6 approximation for a balanced social planner.

7 **6 Generalization**

8 So far, we focused our attention on dynamic clearing games with exponentially
 9 distributed match costs. We shall show below that the developed techniques can
 10 also be used to study a more abstract model. Rather than modeling match costs
 11 directly by defining the distribution of each potential match cost, w_{ij} , we take a
 12 macroscopic viewpoint and posit that the cost of matching a couple depends on
 13 the number of clients and providers currently in the market. This cost may be
 14 the expected cost of matching the cheapest couple or a cost associated to market
 15 making more generally. In practice, this cost can be learned from past data on
 16 clearing events. We shall thus focus our generalization on identifying breaking
 17 points and their respective consequences for different cost regimes.

18 Write $M_C(\tau) = N_C(\tau) - A(\tau)$ for the number of clients in the market at time τ
 19 and $M_P(\tau) = N_P(\tau) - A(\tau)$ for the number of providers respectively. Then, the
 20 expected cost can be written w.l.o.g. as

$$g(M_C, M_P) \tag{39}$$

21 where $g: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-increasing function (in either argument).²⁰
 22 Intuitively, g determines how the expected minimum cost of matching decreases
 23 as more players coexist in the market. For example, if $g(M_C, M_P) = \frac{1}{M_C \cdot M_P}$,
 24 we revert to the previous analysis resulting from exponentially distributed match
 25 costs. This is the case since the expected minimum of $M_C \cdot M_P$ independent $\exp(1)$
 26 random variables is equal to $\frac{1}{M_C \cdot M_P}$.

27 In view of this, it stands to reason that the asymptotic behavior of the market will
 28 be captured by the rate at which the expected minimum matching cost $g(M_C, M_P)$
 29 vanishes as a function of $M_C, M_P \rightarrow \infty$. [Theorem 6](#) below makes this intuition

²⁰We define the function g on the real numbers, but note that it is only the values on $\mathbb{N} \times \mathbb{N}$ which enter the analysis of clearing schedules.

1 precise and identifies a specific threshold beyond which it *is* possible to get a ‘free
 2 lunch’. We restrict our analysis to the case $\delta > 1$ to guarantee that the patient
 3 clearing schedule has finite expected matching cost, i.e., $\text{cost}_{\text{patient}} < \infty$.

4 **Theorem 6.** *Suppose that the expected minimum matching cost decays as $g(\Theta(x), \Theta(x)) =$
 5 $\Theta(1/x^\delta)$ for some $\delta > 1$. Then:*

6 (i) *For $1 < \delta \leq 2$ there is no ‘free lunch’. In particular, the critical rate*
 7 *clearing schedule, that is, the clearing schedule with expected matching*
 8 *ratio $\Theta(\log(A))$, is given by $\text{CS}_{\gamma=1/\delta}$.*

9 (ii) *For $\delta > 2$, ‘free lunch’ exists. In particular, the clearing schedules CS_γ*
 10 *with $\gamma \in (\frac{1}{\delta}, \frac{1}{2}]$ guarantee that the expected matching and waiting ratios*
 11 *are both finite.*

Proof of Theorem 6. We first consider the upper bound. Given $g(\Theta(x), \Theta(x)) =$
 $\Theta(\frac{1}{x^\delta})$ and since g is increasing in both arguments we have:

$$\begin{aligned} \mathbb{E} \left[\sum_{k=1}^A g(M_{\mathcal{C}}, M_{\mathcal{P}}) \mid \min\{M_{\mathcal{C}}, M_{\mathcal{P}}\} = \lfloor k^{1/\delta} \rfloor \right] &\leq \sum_{k=1}^A g(\lfloor k^{1/\delta} \rfloor, \lfloor k^{1/\delta} \rfloor) \\ &= \Theta \left(\sum_{k=1}^A \frac{1}{k} \right) = \Theta(\log A) \quad (40) \end{aligned}$$

12 where the last inequality follows from the bounds for the harmonic series. Thus,
 13 given the optimal clearing schedule has finite matching cost, the expected matching
 14 ratio is smaller than $O(\log A + 1)$.

15 For the lower bound note that $\mathbb{E}[t(k, k^{1/\delta})] < 10k$ by similar arguments as in
 16 Eq. (58) and by recalling that $\delta > 1$. Thus, combining Jensen’s inequality ($\frac{1}{x}$ is
 17 convex) with Markov’s inequality, Lemma 9, and recalling that $\delta \leq 2$ we get:

$$\begin{aligned}
& \sum_{k=1}^A \mathbb{E}[g(M_{\mathcal{C}}, M_{\mathcal{P}}) \mid \min\{M_{\mathcal{C}}, M_{\mathcal{P}}\} = \lfloor k^{1/\delta} \rfloor] \\
&= \sum_{k=1}^A \mathbb{E}[g(M_{\mathcal{C}}, M_{\mathcal{P}}) \mid \min\{M_{\mathcal{C}}, M_{\mathcal{P}}\} = \lfloor k^{1/\delta} \rfloor, t(k, k^{1/\delta}) < 20k] \cdot \mathbb{P}[t(k, k^{1/\delta}) < 20k] \\
&+ \sum_{k=1}^A \mathbb{E}[g(M_{\mathcal{C}}, M_{\mathcal{P}}) \mid \min\{M_{\mathcal{C}}, M_{\mathcal{P}}\} = \lfloor k^{1/\delta} \rfloor, t(k, k^{1/\delta}) > 20k] \cdot \mathbb{P}[t(k, k^{1/\delta}) > 20k] \\
&\geq \sum_{k=1}^A \mathbb{E}[g(M_{\mathcal{C}}, M_{\mathcal{P}}) \mid \min\{M_{\mathcal{C}}, M_{\mathcal{P}}\} = \lfloor k^{1/\delta} \rfloor, t(k, k^{1/\delta}) < 20k] \cdot \frac{1}{2} \\
&= \Theta\left(\sum_{k=1}^A g(\lfloor k^{1/\delta} \rfloor, \lfloor k^{1/\delta} \rfloor)\right) = \Theta\left(\sum_{k=1}^A \frac{1}{k}\right) = \Theta(\log A) \tag{41}
\end{aligned}$$

1 where we used that $M_{\mathcal{C}} = \Theta(M_{\mathcal{P}})$ given $|S_t| = |N_{\mathcal{C}}(t) - N_{\mathcal{P}}(t)| = |M_{\mathcal{C}}(t) - M_{\mathcal{P}}(t)| =$
2 $O(\sqrt{20k})$ since we are in the case $t(k, k^{1/\delta}) < 20k$ and by the assumption that for
3 all λ, μ we have $g(\lambda \cdot x, \mu \cdot x) = \Theta(x^\delta)$. Thus, given the optimal clearing schedule
4 has finite matching cost, the expected matching ratio is $\Omega(\log A)$, concluding the
5 proof together with the upper bound.

6 To summarize, for $\delta \in (1, 2]$ the critical rate clearing schedule is given by $\mathbf{CS}_{\gamma=1/\delta}$.
7 Thus the critical rate clearing schedules are given by \mathbf{CS}_{γ} with $\gamma \in (0, 1/2]$ and by
8 [Theorem 4](#), the expected waiting ratio for these schedules is not finite. Note that
9 the case $\delta = 2$ is simply [Theorem 2](#). We conclude that there is no ‘free lunch’.

10 (2) By [Theorem 4](#) the expected waiting ratio is finite for all clearing schedules \mathbf{CS}_{γ}
11 with $\gamma \leq \frac{1}{2}$.

We upper bound the expected matching ratio for the clearing schedule \mathbf{CS}_{γ} for
 $\gamma > \frac{1}{\delta}$: For the upper bound we have with $g(\Theta(x), \Theta(x)) = \Theta\left(\frac{1}{x^\delta}\right)$:

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^A g(M_{\mathcal{C}}, M_{\mathcal{P}}) \mid \min\{M_{\mathcal{C}}, M_{\mathcal{P}}\} = \lfloor k^{1/\delta} \rfloor\right] &\leq \sum_{k=1}^A g(\lfloor k^{1/\delta} \rfloor, \lfloor k^{1/\delta} \rfloor) \\
&= \Theta\left(\sum_{k=1}^A \frac{1}{k^{\gamma \cdot \delta}}\right) = \Theta(1) \tag{42}
\end{aligned}$$

12 where the last identity holds since $\gamma > \frac{1}{\delta}$.

13 Thus, for given δ the clearing schedules \mathbf{CS}_{γ} with $\gamma \in (\frac{1}{\delta}, \frac{1}{2}]$ guarantee that the
14 expected matching ratio and waiting ratio are both finite, i.e., free lunch. \square

1 **Theorem 6(i)** extends our previous analysis by showing how the critical rate clear-
 2 ing schedule moves dependent on δ . In fact **Theorem 6(ii)** shows that the con-
 3 clusion of **Theorem 2** does not hold for the regime $\delta > 2$, that is, there exists
 4 a free lunch and in particular it is achieved by the clearing schedules CS_γ with
 5 $\gamma \in (\frac{1}{8}, \frac{1}{2}]$. This is because for quickly decaying matching costs it becomes easier
 6 to choose a ‘good’ schedule, and thus the ‘window of opportunity’ is increasing in
 7 the derivative of g . One can build intuition for this result by reasoning about mar-
 8 ket settings that differ in terms of match cost variability: in markets where match
 9 costs are generally rather similar, thickening the market by waiting will only lead
 10 to a meaningful positive effect in terms of expected match cost reduction when
 11 waiting for a long time. By contrast, when match costs vary substantially, match
 12 costs reduce in expectation with much less delay, thus making it more likely for a
 13 mechanism designer to get a free lunch. Importantly, the clearing schedule $\text{CS}_{\text{greedy}}$
 14 is never optimal when dealing with many types, independent of the match cost
 15 distribution at hand.

16 7 Discussion

17 In this paper, we studied the *dynamic clearing game*, where heterogeneous clients
 18 and providers arrive uncoordinatedly in order to be matched. We studied the
 19 trade-off a social planner is facing between two competing objectives: a) to reduce
 20 players’ *waiting time* before getting matched; and b) to form efficient pairs in
 21 order to reduce *matching cost*.

22 Our analysis of the dynamic clearing game reveals that a multi-objective social
 23 planner often faces a substantial trade-off. Starting with the micro-founded model
 24 for match costs we showed that there exists no free lunch, that is, there is no
 25 clearing schedule that is approximately optimal in terms of both waiting time and
 26 matching cost. We identified a unique breaking point where a stark reduction in
 27 matching cost compared to a stark increase in waiting cost occurs. In line with
 28 recent works by Ashlagi et al. (2017), Ashlagi et al. (2018) and many others, we
 29 focused on a concrete class of social welfare functions that weigh costs from waiting
 30 versus matching on a comparable scale and identify the optimal clearing schedule,
 31 namely, the clearing schedule that matches the k -th couple when $\Theta(\sqrt{k}(\log k)^{1/3})$
 32 players are on the short side of the market.

33 Generalizing the model, we abstract away from modeling match costs directly
 34 and take a macroscopic viewpoint. Positing that the cost of matching a couple

1 depends on the number of clients and providers currently in the market we identify
2 two regimes. One, where no free lunch continues to hold, the other, where there
3 is a window of opportunity to be optimal along both dimensions, that is free
4 lunch.

5 There are multiple directions in which our analysis could be extended. Perhaps the
6 most evident avenue for future research is to model market participation behavior
7 game-theoretically, which would lead to new strategic considerations and probably
8 induce other matchings (see, e.g., [Baccara et al. 2018](#)). This analysis could be
9 pursued in more applied contexts, for instance relating to our motivating example
10 of a labor market with a central employment bureau, where waiting costs could
11 be interpreted as benefits payable by the bureau. An unemployed worker might
12 forgo some of these benefits by (repeatedly) rejecting matches. This is the case
13 because longer waiting, even though borne out of strategic behavior, may improve
14 the match quality (reducing matching cost).

15 A second route for further investigation is to enlarge the options of the social
16 planner in terms of clearing schedules. For one, the social planner could be learning
17 from market observations about the distribution of match costs, which incidentally
18 we may also allow to follow other, more general classes of distributions. This would
19 allow the social planner to formulate more sophisticated clearing schedules that
20 incorporate match costs between players that are currently in the market. In
21 particular, if the social planner learns that a given agent may be ‘hard to match’,
22 then it might be sensible to match that agent directly and not incur further waiting
23 cost. Furthermore, the social planner may want to match more than one couple
24 at a time.

25 The study of dynamic market institutions is clearly fascinating, with tremendous
26 scope for progress in (old and new) applications, where research has only just
27 started. Our contribution has been to go beyond binary match values, and to iden-
28 tify breaking points under incomplete information. We hope that our framework
29 is able to provide fertile ground for further research, both theoretical and applied
30 to real-world market contexts, in particular as regards thinking about whether
31 the kinds of breaking points we describe are relevant in the optimal design of such
32 markets.

1 Appendix

2 A Proof of Theorem 3

3 Before turning to the proof we introduce the following definition and lemmas.

4 **Definition 7.** Let X_1, X_2, \dots be iid random variables with $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i =$
5 $-1] = \frac{1}{2}$.

6 • Let $S_k = \sum_{i=1}^k X_i$.

7 • Let $t(k, C)$ be the stopping time for the event that for the k -th time at least
8 C clients and C providers are in the market, assuming that every time this
9 is the case one client and one provider are removed.

10 **Lemma 8.** Let $w_{ij} \sim \exp(\lambda_j)$ for $j = 1, 2, \dots, N$, be a family of independent
11 exponentially distributed random variables. Then

$$\min\{w_{i1}, w_{i2}, \dots, w_{iN}\} \sim \exp\left(\sum_{j=1}^N \lambda_j\right). \quad (43)$$

12 In particular, if for all j , $\lambda_j = 1$, then $\mathbb{E}[\min_j w_{ij}] = \frac{1}{N}$.

13 *Proof.* This proof is standard but we repeat it for the sake of completeness. The
14 random variable w_{ij} has cumulative distribution function

$$F_{w_{ij}} = \mathbb{P}(w_{ij} \leq x) = 1 - e^{-\lambda_j x} \quad \text{for all } x > 0 \text{ and all } j = 1, 2, \dots, N. \quad (44)$$

Now, define the random variable $Y = \min\{w_{i1}, w_{i2}, \dots, w_{iN}\}$. Then, the cumulative distribution function of Y is

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= 1 - \mathbb{P}(Y \geq y) \\ &= 1 - \mathbb{P}(\min\{w_{i1}, w_{i2}, \dots, w_{iN}\} \geq y) \\ &= 1 - \mathbb{P}(w_{i1} \geq y) \cdot \mathbb{P}(w_{i2} \geq y) \cdot \dots \cdot \mathbb{P}(w_{iN} \geq y) \\ &= 1 - e^{-\lambda_1 y} \cdot e^{-\lambda_2 y} \cdot \dots \cdot e^{-\lambda_N y} \\ &= 1 - e^{-\sum_{j=1}^N \lambda_j y} \quad y > 0 \end{aligned} \quad (45)$$

15 The latter cumulative distribution function is that of an exponential variable with
16 parameter $\sum_{j=1}^N \lambda_j$. \square

1 **Lemma 9.** For S_k defined as above we have:²¹

$$0.67 \cdot \sqrt{k} \lesssim \frac{2\pi}{e^2} \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{k} \leq \mathbb{E}[|S_k|] \leq \frac{e}{\sqrt{\pi}} \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{k} \lesssim 1.23 \cdot \sqrt{k} \quad (46)$$

Proof. The starting point of our proof is an intermediate result in the proof of the limit of the expected absolute value of the 1-d random walk, which is detailed in Hizak and Logozar (2011, Equations 29a and 29b) and is based on combinatorial arguments via the binomial distribution:

$$\mathbb{E}[|S_k|] = \begin{cases} \frac{1}{2^{k-2}} \frac{k}{2} \binom{k-1}{k/2} = \frac{k}{2^k} \frac{k!}{[(k/2)!]^2} & \text{for } k \text{ even,} \\ \frac{1}{2^{k-1}} \frac{k+1}{2} \binom{k}{(k+1)/2} = \frac{k+1}{2^{k+1}} \frac{(k+1)!}{[((k+1)/2)!]^2} & \text{for } k \text{ odd.} \end{cases} \quad (47)$$

2 Since $\mathbb{E}[|S_{2k}|] = \mathbb{E}[|S_{2k-1}|]$ it suffices to analyze the case where k is even. To that
3 end, we will use Stirling's formula to bound $k!$ from above and below as

$$\sqrt{2\pi} \cdot k^{k+1/2} \cdot e^{-k} \leq k! \leq e \cdot k^{k+1/2} \cdot e^{-k} \quad (48)$$

For k even, we may bound $|S_k|$ from above as:

$$\mathbb{E}[|S_k|] = \frac{k}{2^k} \frac{k!}{[(k/2)!]^2} \leq \frac{k}{2^k} \frac{e \cdot k^{k+1/2} \cdot e^{-k}}{2\pi \cdot (k/2)^{k+1} \cdot e^{-k}} = \frac{e}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{k} \quad (49)$$

Next, we lower bound $|S_k|$ for k even:

$$\mathbb{E}[|S_k|] = \frac{k}{2^k} \frac{k!}{[(k/2)!]^2} \geq \frac{k}{2^k} \frac{\sqrt{2\pi} \cdot k^{k+1/2} \cdot e^{-k}}{e^2 \cdot (k/2)^{k+1} \cdot e^{-k}} = \frac{2\pi}{e^2} \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{k} \quad (50)$$

This concludes the proof for k even. For k odd we have with the observation that $|S_k| = |S_{k+1}|$:

$$\frac{2\pi}{e^2} \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{k} \leq \frac{2\pi}{e^2} \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{2\lceil k/2 \rceil} \leq \mathbb{E}[|S_{k+1}|] = \mathbb{E}[|S_k|] \quad (51a)$$

and

$$\frac{e}{\sqrt{\pi}} \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{k} \geq \frac{e}{\sqrt{\pi}} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{2\lceil k/2 \rceil} \geq \mathbb{E}[|S_{k+1}|] = \mathbb{E}[|S_k|] \quad (51b)$$

²¹Note that $\lim_{k \rightarrow \infty} \mathbb{E}[|S_k|] = \sqrt{\frac{2}{\pi}} \cdot \sqrt{k}$ (Peters, 1856).

2 For the sake of limiting notation the proposition and proof are stated for the
 3 clearing schedules with $f(k) = \lceil k^\gamma \rceil$ rather than for $\Theta(f)$. Adding constant upper
 4 and lower bounds is straightforward and thus omitted. Recall that $t(k, f(k))$ is
 5 the stopping time for the event that for the k -th time at least $f(k)$ clients and
 6 $f(k)$ providers are in the market, assuming that every time this is the case one
 7 client and one provider are removed.

8 *Proof of Theorem 3.* Throughout the proof we shall simplify notation by omitting
 9 the fact that some of the matching schedules are defined via the ceiling of functions
 10 mapping to \mathbb{R}^+ (e.g., $\lceil k^\gamma \rceil$). The results are not changed by the omission since
 11 match costs are never underestimated and overestimated by very little. Further,
 12 while they are stated together in the proposition, we study the clearing schedules
 13 $\text{CS}_\gamma = 0$ and $\text{CS}_{0 < \gamma < 1/2}$ separately since they require different arguments.

14 **First come, first served** (CS_{FCFS}) In FCFS the cost of each match is the ex-
 15 pected of a single match cost, that is λ . After A matches have occurred, the
 16 expected incurred cost is λA . Thus, given the patient clearing schedule has cost
 17 bounded above by $\frac{\pi^2}{6\lambda}$, the expected matching ratio is equal to $\frac{\lambda \cdot A}{\pi^2/(6\lambda)}$.

18 Before stating the proofs for the other results recall from the proof of [Proposition 1](#),
 19 that the exponential distribution is closed under scaling. We shall thus simplify
 20 notation and assume that for all i, j $w_{ij} \sim \exp(1)$. Note that, for lower bounds
 21 the scaling factor $\frac{1}{\lambda}$ needs to be applied and for upper bounds the scaling $\frac{1}{\lambda}$ needs
 22 to be applied. But note that those scaling factors are constant with respect to τ
 23 (and thus A) and therefore do not influence the orders of the limiting results.

24 **Greedy matching** ($\text{CS}_{\text{greedy}}$) The k -th match happens when the minimum of
 25 the number of clients and providers who already arrived to the market is k , that
 26 is, at time $t(k, 1)$. The expected weight of the k -th match depends on the number
 27 of players currently present on the long side of the market (since on the short
 28 side there is only one agent). This random variable is given by $|S_{t(k,1)}| + 1$. By
 29 [Lemma 8](#) the expected weight thus is $\mathbb{E}[\frac{1}{|S_{t(k,1)}| + 1}]$. The first A matches thus have
 30 an expected cost of

$$\mathbb{E}\left[\sum_{k=1}^A \frac{1}{|S_{t(k,1)}| + 1}\right]. \quad (52)$$

1 Given that we study fixed A (the number of matches that) we have:²²

$$\mathbb{E}\left[\sum_{k=1}^A \frac{1}{|S_{t(k,1)}| + 1}\right] = \sum_{k=1}^A \mathbb{E}\left[\frac{1}{|S_{t(k,1)}| + 1}\right] \quad (53)$$

2 Next, by Jensen's inequality ($\frac{1}{x}$ is convex) we have:

$$\sum_{k=1}^A \mathbb{E}\left[\frac{1}{|S_{t(k,1)}| + 1}\right] > \sum_{k=1}^A \frac{1}{\mathbb{E}[|S_{t(k,1)}| + 1]} = \sum_{k=1}^A \frac{1}{\mathbb{E}[|S_{t(k,1)}|] + 1} \quad (54)$$

3 We shall now approximate $\mathbb{E}[t(k, 1)]$. By Lemma 9 we have $\mathbb{E}[S_t] < 1.23\sqrt{t}$. Thus
 4 the short side of the market has $\frac{t-1.23\sqrt{t}}{2}$ agents. Setting $k = \frac{t-1.23\sqrt{t}}{2}$ and solving
 5 the quadratic equation we find the crude upper bound for the expectation:²³

$$\mathbb{E}[t(k, 1)] = \left(\frac{1.23 + \sqrt{1.23^2 + 8k}}{2}\right)^2 < \frac{3}{4} + 2k + 2\sqrt{k} < 5k \quad (55)$$

Returning to Eq. (54) we have with Lemma 9:

$$\sum_{k=1}^A \frac{1}{\mathbb{E}[|S_{t(k,1)}|] + 1} > \sum_{k=1}^A \frac{1}{\mathbb{E}[|S_{5k}|] + 1} > \frac{1}{1.23} \sum_{k=1}^A \frac{1}{\sqrt{5k} + 1} \quad (56)$$

$$> \frac{1}{1.23} A \cdot \frac{1}{\sqrt{5A} + 1} > A \cdot \frac{1}{5\sqrt{A}} = \frac{\sqrt{A}}{5} \quad (57)$$

6 Thus, given the optimal schedule has $\text{cost}_{\text{patient}}(A) \leq \frac{\pi^2}{6\lambda}$, the expected matching
 7 ratio is lower bounded by $\frac{\sqrt{A}}{5\pi^2/(6\lambda)}$.

8 **Subcritical matching** ($\text{CS}_{\gamma=0}$) We shall fix the clearing schedule such that it
 9 matches a couple every time some fixed $C \in \mathbb{N}$ players are on the short side of the
 10 market ($N - A = C$) and note that it belongs to the family of clearing schedules
 11 $\text{CS}_{\gamma=0}$.

12 Next, note that $t(k, C) = t(1, C) + t(k - 1, 1)$, since we assume that every time

²²Note that t (the total number of client and providers who have arrived to the market) depends on A (and vice versa). Therefore, Wald (1944)'s equation does not apply and thus the route of inquiry to study the matching cost at some continuous time τ does not work since we could not interchange summation and expectation.

²³We solve $k = \frac{t-1.23\sqrt{t}}{2}$. Setting $t = u^2$ and rearranging we solve quadratic equation $u^2 - u - 2k \stackrel{!}{=} 0$. The solutions are:

$$u_{1,2} = \frac{1.23 \pm \sqrt{1.23^2 + 8k}}{2}$$

Given the variable transformation the positive solution is selected.

1 at least C clients and C providers are in the market exactly one client and
 2 one provider match and thus leave the market. Similarly to the proof of [The-](#)
 3 [orem 3](#) ($\text{CS}_{\text{greedy}}$) we can bound $\mathbb{E}[t(1, C)]$ from above by noting that it is equal to
 4 $\mathbb{E}[t(C, 1)]$. Thus

$$\mathbb{E}[t(k, C)] = \mathbb{E}[t(C, 1)] + \mathbb{E}[t(k-1, 1)] < 5C + 5k \quad (58)$$

With the latter and, as above by Jensen's inequality ($\frac{1}{x}$ is convex) and [Lemma 9](#) we have for the expected matching cost:

$$\begin{aligned} \mathbb{E}\left[\sum_{k=1}^A \frac{1}{(C + |S_{t(k,C)}|)C}\right] &\geq \sum_{k=1}^A \frac{1}{(C + \mathbb{E}[|S_{t(k,C)}|])C} \geq \sum_{k=1}^A \frac{1}{(C + \mathbb{E}[|S_{5C+5k}|])C} \\ &\geq \frac{1}{1.23} \sum_{k=1}^A \frac{1}{(C + \sqrt{5C + 5k})C} \geq \frac{A}{1.23} \cdot \frac{1}{(C + \sqrt{5C + 5A})C} \\ &= \frac{1}{1.23 \cdot C} \cdot \frac{A - C}{\sqrt{5C + 5A} + C} = \Omega(\sqrt{A}) \end{aligned} \quad (59)$$

5 Thus, given the optimal clearing schedule has finite cost the expected matching
 6 ratio is $\alpha(A) = \Omega(\sqrt{A})$.

7 The second part of the assertion follows by observing:

$$\mathbb{E}\left[\sum_{k=1}^A \frac{1}{(C + |S_{t(k,C)}|)C}\right] < \frac{1}{C} \mathbb{E}\left[\sum_{k=1}^A \frac{1}{1 + |S_{t(k,1)}|}\right] \quad (60)$$

Subcritical matching ($\text{CS}_{0 < \gamma < 1/2}$) For the upper bound, by [Lemma 8](#), we have:

$$\begin{aligned} \mathbb{E}\left[\sum_{k=1}^A \frac{1}{(k^\gamma + |S_{t(k,\sqrt{k})}|)k^\gamma}\right] &\leq \sum_{k=1}^A \frac{1}{k^{2\gamma}} = 1 + \sum_{k=2}^A \frac{1}{k^{2\gamma}} \leq 1 + \int_{x=1}^A \frac{1}{x^{2\gamma}} dx \\ &= 1 + \left[\frac{1}{1-2\gamma} x^{1-2\gamma}\right]_{x=1}^A \leq 1 + \frac{1}{1-2\gamma} A^{1-2\gamma} \end{aligned} \quad (61)$$

8 Thus, given the optimal clearing schedule has finite cost, the expected matching
 9 ratio is $\alpha(A) = \mathcal{O}(A^{1-2\gamma})$ for $0 < \gamma < \frac{1}{2}$.

For the lower bound, note that $t(k, k^\gamma) < 10k$ for $\gamma < 1$ by similar arguments as in [Eq. \(58\)](#). Further note that $\mathbb{E}[|S_t|]$ is strictly increasing in t . Thus, with Jensen's

inequality ($\frac{1}{x}$ is convex):

$$\begin{aligned}
\sum_{k=1}^A \mathbb{E}\left[\frac{1}{(k^\gamma + |S_{t(k,k^\gamma)}|)k^\gamma}\right] &> \sum_{k=1}^A \frac{1}{(k^\gamma + \mathbb{E}[|S_{10k}|])k^\gamma} > \frac{1}{1.23} \sum_{k=1}^A \frac{1}{(k^\gamma + \sqrt{10k})k^\gamma} \\
&> \frac{1}{1.23(\sqrt{10} + 1)} \sum_{k=1}^A \frac{1}{k^{\frac{1}{2} + \gamma}} > \frac{1}{6} \int_{x=1}^A \frac{1}{x^{\frac{1}{2} + \gamma}} dx \\
&> \frac{1}{6} \left[\frac{1}{\frac{1}{2} - \gamma} x^{\frac{1}{2} - \gamma} \right]_{x=1}^A = \Omega(A^{\frac{1}{2} - \gamma}) \tag{62}
\end{aligned}$$

1 **Critical matching** ($\text{CS}_{\gamma=1/2}$) For the upper bound, by Lemma 8, we have:

$$\mathbb{E}\left[\sum_{k=1}^A \frac{1}{(\sqrt{k} + |S_{t(k,\sqrt{k})}|)\sqrt{k}}\right] < \sum_{k=1}^A \frac{1}{k} \leq \log A + 1 \tag{63}$$

2 where the last inequality follows from the bounds for the harmonic series. Thus,
3 given the optimal clearing schedule has finite matching cost (lower bounded by
4 $\frac{\log(2)}{\lambda}$), the expected matching ratio is smaller than $\frac{\frac{1}{\lambda}(\log A + 1)}{\log(2)/\lambda}$.

For the lower bound note that $\mathbb{E}[t(k, \sqrt{k})] < 10k$ by similar arguments as in Eq. (58). Further note that S_t is strictly increasing in t . Thus, with Jensen's inequality ($\frac{1}{x}$ is convex) and Lemma 9:

$$\begin{aligned}
\sum_{k=1}^A \mathbb{E}\left[\frac{1}{(\sqrt{k} + |S_{t(k,\sqrt{k})}|)\sqrt{k}}\right] &> \sum_{k=1}^A \frac{1}{(\sqrt{k} + \mathbb{E}[|S_{10k}|])\sqrt{k}} > \frac{1}{1.23} \sum_{k=1}^A \frac{1}{(\sqrt{k} + \sqrt{10k})\sqrt{k}} \\
&\geq \frac{1}{6} \sum_{k=1}^A \frac{1}{k} > \frac{1}{6} \log A \tag{64}
\end{aligned}$$

5 Thus, given the optimal clearing schedule has $\text{cost}_{\text{patient}}(A) \leq \frac{\pi^2}{6\lambda}$, the expected
6 matching ratio is bounded below by $\frac{\frac{1}{6} \log A}{\frac{\pi^2}{6\lambda}} = \frac{\lambda}{\pi^2} \cdot \log A$.

7 **Supercritical matching** ($\text{CS}_{1/2 < \gamma \leq 1}$) As above, by Jensen's inequality (since
8 $1/x$ is convex) and Lemma 8 we have:

$$\mathbb{E}\left[\sum_{k=1}^A \frac{1}{(k^\gamma + |S_{t(k,k^\gamma)}|)k^\gamma}\right] < \sum_{k=1}^A \mathbb{E}\left[\frac{1}{k^\gamma k^\gamma}\right] = \sum_{k=1}^A \frac{1}{k^{2\gamma}} \rightarrow \zeta(2\gamma) \tag{65}$$

9 where ζ is the Riemann zeta function and is known to converge for $\gamma > \frac{1}{2}$. Given
10 that we are considering a sum with positive summands convergence is from below.

1 Thus, given the optimal clearing schedule has finitematching cost ($\text{cost}_{\text{patient}}(A) \geq$
2 $\frac{\log(2)}{\lambda}$), the expected matching ratio is bounded from above by $(\bar{\lambda}/\lambda) \cdot \zeta(2\gamma)/\log 2$
3 for $\frac{1}{2} < \gamma \leq 1$. □

4 B Proof of Theorem 4

5 *Proof of Theorem 4.* First note that in order to compare different clearing sched-
6 ules we are interested in the additional waiting time incurred until some number A
7 of couples have been matched. Thus, we consider the waiting time until T for the
8 greedy schedule (the benchmark) and for other schedules the waiting time until \hat{T}
9 where \hat{T} is the expected time until under the given schedule the same number of
10 couples have been matched as in the greedy schedule until time T .

11 As in the Proof of Theorem 3 we shall simplify notation by omitting the fact that
12 some of the matching schedules are defined via the ceiling function of functions
13 mapping to \mathbb{R}^+ (e.g., $\lceil k^\gamma \rceil$). We invite the reader to convince her-or himself that
14 the results are not altered through this simplification.

15 Let $\tau(k)$ be the moment the k -th couple is matched (given a particular clearing
16 schedule). We proceed in a case-by-case basis below:

17 **First come, first served (CS_{FCFS})** It suffices to note that this clearing schedules
18 matches players at exactly the same moments as the greedy clearing schedules.
19 The result then follows.

20 **Subcritical and critical matching (CS<sub>0<math>\leq \gamma \leq 1/2 We shall study the worst case
21 such clearing schedule with respect to waiting time. We consider two different
22 parts. In the first part we wait until at least T^γ clients and T^γ providers are in the
23 market. The second part then proceeds in the same way as the greedy clearing
24 schedule, keeping in mind that at all future times $\min\{N_C, N_P\}$ is exactly T^γ . The
25 expected waiting time of the first schedule can be bounded above by the upper
26 bound for the expected time until T^γ clients and T^γ providers are in the market,
27 that is, $\mathbb{E}[\tau(5T^\gamma)] = 5T^\gamma$ (see Eq. (55) in the proof of Theorem 3) noting that we
28 used the fact that the arrival of agents is governed by a Poisson clock of rate 1.
29 Now, a crude upper bound for the waiting time of the first part of the process is
30 found be assuming that all agents are in the market from the beginning ($\tau = 0$),
31 yielding the upper bound $5T^\gamma \cdot 5T^\gamma$.</sub>**

32 Note that, the first part of the process takes $\hat{T} - T$ time. For the remaining second

1 part of the process the waiting cost is the cost of the greedy schedule ($\frac{2}{3}T^{3/2}$) plus
 2 the cost of the – in expectation – no more than $5T^\gamma$ agents on each side of the
 3 market to ‘remain’ for the subsequent periods. Thus the total waiting time is
 4 bounded above by:

$$5T^\gamma \cdot 5T^\gamma + \frac{2}{3}T^{3/2} + 5T^\gamma \cdot T = \Theta(T^{3/2}) \quad (66)$$

5 Thus $\beta(\hat{T}) = (3/2)\Theta(T^{3/2})/T^{3/2} = \Theta(1)$.

6 **Supercritical matching** ($\text{CS}_{1/2 < \gamma \leq 1}$) We first construct a lower bound. Con-
 7 sider the alternative arrival process, where clients and providers alternately ar-
 8 rive to the market. Note that for any given clearing schedule this process incurs
 9 lower waiting time. For the clearing schedule we consider the waiting time of this
 10 alternative arrival process is precisely governed by the fact that the k -th match
 11 takes place when at least k^γ players are on the short side of the market. Further
 12 note that $\hat{T} \geq T$. Thus, the waiting time is lower bounded by using the approxi-
 13 mation by the Wiener process (by arguments as in [Proposition 1](#) and by observing
 14 that arrival is governed by a Poisson clock of rate 1):

$$\int_0^T 2\tau^\gamma d\tau = \frac{2}{1+\gamma}\tau^{1+\gamma}\Big|_0^T = \Omega(T^{1+\gamma}) \quad (67)$$

15 For the upper bound, we construct a clearing schedule that constitutes an upper
 16 bound of the schedule under consideration. For fixed k , let $T = \tau(k)$ consider
 17 the following clearing schedule: First wait until there are at least k^γ clients and
 18 providers in the market, then proceed with the greedy schedule such that at any
 19 future point $\min\{\text{clients}, \text{providers}\}$ in the market is equal to k^γ . Note that this
 20 new schedule has the same total run time as the original schedule, that is, \hat{T} .
 21 Further it is evident that the waiting time occurred by the new schedule is greater
 22 than the waiting time of the original schedule. By arguments as for ($\text{CS}_{\gamma=0}$) and
 23 by the fact that arrival is governed by a Poisson clock of rate 1 we can upper
 24 bound the waiting time by:

$$5T^\gamma \cdot 5T^\gamma + \frac{2}{3}T^{3/2} + 5T^\gamma \cdot T = \Theta(T^{1+\gamma}) \quad (68)$$

25 since we assumed $\frac{1}{2} < \gamma \leq 1$.

26 The two bounds together show that the waiting time of the originally considered

1 clearing schedule is $\Theta(T^{1+\gamma})$. Thus $\beta(\hat{T}) = \frac{\Theta(T^{1+\gamma})}{(2/3)T^{3/2}} = \Theta(T^{\gamma-\frac{1}{2}})$.

2 **Patient matching** ($\text{CS}_{\text{patient}}$) First note that for the patient schedule $\hat{T} = T$.
 3 The expected waiting time for the patient schedule until time T is given by

$$\mathbb{E} \left[\int_0^T N_{\mathcal{C}}(\tau) + N_{\mathcal{P}}(\tau) d\tau \right] = \int_0^T \mathbb{E}[N_{\mathcal{C}}(\tau) + N_{\mathcal{P}}(\tau)] d\tau \quad (69)$$

4 where the latter equality holds by Tonelli's theorem (by noting that $N_{\mathcal{C}}(\tau) + N_{\mathcal{P}}(\tau)$
 5 is non-negative). The expectation is with respect to the number of clients and
 6 providers and with respect to the arrival times of the agents (governed by a Poisson
 7 clock). Again by Tonelli's theorem we can consider the case where the expectation
 8 with respect to the arrival times is taken first. Then by the fact that the arrival
 9 of agents is assumed to follow a Poisson clock of rate 1 we have:

$$\int_0^T \mathbb{E}[N_{\mathcal{C}}(\tau) + N_{\mathcal{P}}(\tau)] d\tau = \int_0^T \lfloor \tau \rfloor d\tau = \Theta(T^2) \quad (70)$$

10 Thus $\beta(\hat{T}) = \frac{\Theta(T^2)}{\frac{2}{3}T^{3/2}} = \Theta(\sqrt{T})$. □

11 C Proof of approximation in Proof of Theorem 5

12 *Proof of omitted approximation in Proof of Theorem 5.* We begin by ap-
 13 proximating the two sums in Eq. (31), i.e.,

$$\sum_{k=1}^A \frac{1}{f(k)^2} = \Theta \left(\frac{1}{A^{3/2}} \sum_{k=1}^A f(k) \right) \quad (71)$$

14 Recalling that f is assumed non-decreasing for large k , the summand on the left-
 15 hand side is decreasing and

$$\int_0^A \frac{1}{f(x)^2} dx \geq \sum_{k=1}^A \frac{1}{f(k)^2} \geq \int_1^{A+1} \frac{dx}{f(x)^2} \quad (72)$$

16 Considering the meaning of $f(k)$ it is without loss of generality to define $f(x) = 1$
 17 for $x \in [0, 1)$ since the summand $\frac{1}{f(k)^2}$ remains decreasing. Thus the absolute
 18 difference between the two bounds is bounded above by:

$$\left| \int_0^A \frac{1}{f(x)^2} dx - \int_1^{A+1} \frac{1}{f(x)^2} dx \right| = \left| \int_0^1 \frac{1}{f(x)^2} dx - \int_A^{A+1} \frac{1}{f(x)^2} dx \right| \leq 1 \quad (73)$$

1 It follows that

$$\sum_{k=1}^A \frac{1}{f(k)^2} = \Theta\left(\int_1^{A+1} \frac{1}{f(x)^2} dx\right) \quad (74)$$

2 Next consider the right-hand side of (31). The summand is increasing, so we get:

$$\frac{1}{A^{3/2}} \int_0^A f(x) dx \leq \frac{1}{A^{3/2}} \sum_{k=1}^A f(k) \leq \frac{1}{A^{3/2}} \int_1^{A+1} f(x) dx \quad (75)$$

Now note that $f(x) < x$ must hold. Thus the absolute difference between the two bounds is bounded above by:

$$\begin{aligned} \frac{1}{A^{3/2}} \left| \int_0^A f(x) dx - \int_1^{A+1} f(x) dx \right| &= \frac{1}{A^{3/2}} \left| \int_A^{A+1} f(x) dx - \int_0^1 f(x) dx \right| \\ &\leq \frac{A}{A^{3/2}} = \mathcal{O}(1). \end{aligned} \quad (76)$$

3 It follows that

$$\frac{1}{A^{3/2}} \sum_{k=1}^A f(k) = \Theta\left(\frac{1}{A^{3/2}} \int_1^{A+1} f(x) dx\right) \quad (77)$$

4 With above approximations it follows that Eq. (31) holds if and only if the follow-
5 ing equation holds:

$$\int_1^A \frac{1}{f(x)^2} dx = \Theta\left(\frac{1}{A^{3/2}} \int_1^A f(x) dx\right) \quad (78)$$

6 as claimed. □

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